Minimum Mean Square Error Quantizers

P. Kabal

INRS-Télécommunications
3 Place du Commerce
Ile des Soeurs, Qué.
CANADA H3E 1H6

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ABSTRACT

This report discusses the design of quantizers which minimize the mean square error for a signal with a given probability density function. Tables of optimal non-uniform quantizers are given for signals with Gaussian, Laplace (exponential) and gamma distributions. These figures correct values given previously in the literature. An appendix documents a program for calculating an optimal quantizer for an empirically derived tabulated probability density.

MINIMUM MEAN SQUARE ERROR QUANTIZERS

Digital transmission of signals plays an important role in telecommunications. The process of converting an analogue signal into a digital form is inherent in many of these systems. The discretization of the analogue information proceeds in two steps. The signal is sampled and then the continuum of sample values is represented by a finite set of levels. The sampling process is theoretically information lossless if the input signal is bandlimited and if it is sampled at a rate which exceeds twice its bandwidth.

The second operation, quantization of the sample values, introduces inevitable errors in the reproduced signal. This report is concerned with the design of quantizers which minimize the effects of these errors.

Quantization is the process of subdividing the range of a signal into non-overlapping regions. An output level is then assigned to each region. Since this output level is used to represent all of the values in the region, it is usually itself within that region (see Fig. 1). The quantizer as defined here is a memoryless device.

Consider a N level quantizer, with output levels $y_1, y_2, \ldots, y_N$.
Let the input regions be

$$R_i = \{x_{i-1} < x < x_i\} , \quad i = 1,2,\ldots,N$$

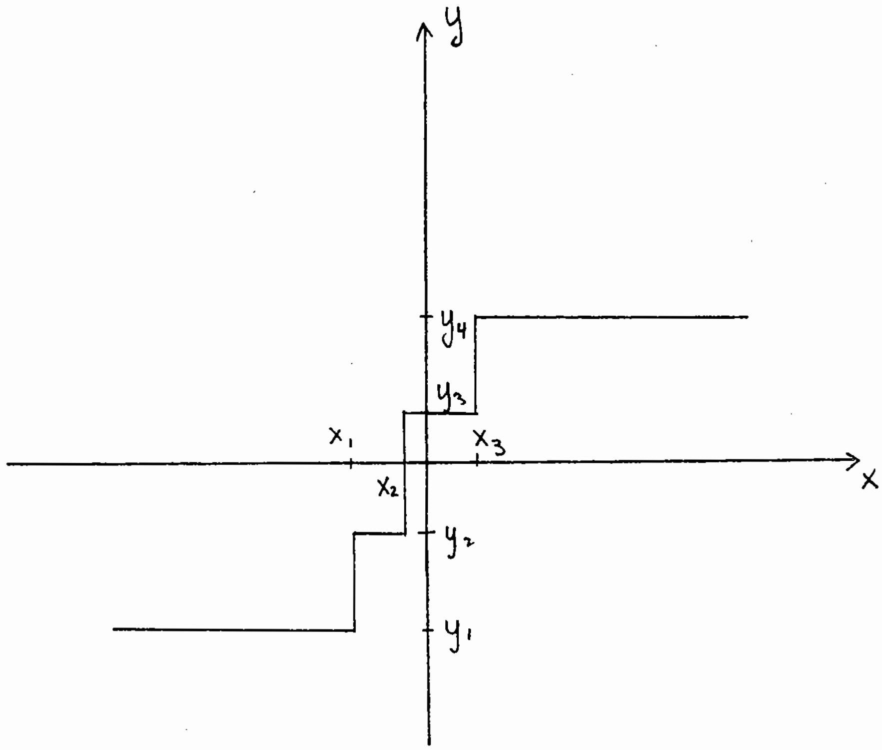

Fig. 1 Quantizer Characteristics

For convenience the x_i are in increasing order, and $x_0 = -\infty$ and $x_N = \infty$. The difference between the input signal and its quantized version is the quantization error,

$$e = x - y_k, \text{ for } k \text{ such that } x \in R_k$$

The quantization noise is often categorized into two parts. The overload component occurs when the input signal greatly exceeds the range of the quantizer output levels, i.e.

$$x < y_1, \quad \text{or} \quad x > y_N$$

The granular noise is that due to representing the continuum of input values by a finite number of output levels. The overload is reduced by choosing a larger range for the quantizer. The granular noise is reduced by choosing more closely spaced quantization levels. Thus for a fixed number of levels N, the combined effect of overload and granular noise is minimized by some compromise between them. Stated another way, the range of the quantizer must be matched to the signal to be quantized.

Necessary Conditions for Error Minimization

Assume the error criterion is the average value of g(e) where g(.) is some differentiable function. Then we wish to minimize

$$E[g(e)] = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} g(x-y_i)\, p(x)\, dx \tag{1}$$

Only the probability density function of x enters into the formulation since the quantizer was assumed to be memoryless.

Differentiating (1) with respect to x_k and y_k gives the following

$$g(x_k-y_k)\, p(x_k) - g(x_k-y_{k+1})\, p(x_k) = 0 \qquad k = 1,2,\ldots,N-1 \tag{2}$$

$$\int_{x_{k-1}}^{x_k} g'(x-y_k)\, p(x)\, dx = 0, \quad k = 1, 2,\ldots,N. \tag{3}$$

Equation (2) can be simplified if

$p(x_k) \neq 0$

$g(a) = g(-a)$ and

$g(a_1) \geq g(a_0)$ for $0 \leq a_0 \leq a_1$

Thus we are assuming g(.) is an even, montotonically increasing function of its argument. Then (2) becomes

$$|x_k - y_k| = |x_k - y_{k+1}| \qquad \text{or}$$

$$x_k = \frac{y_k + y_{k+1}}{2}, \quad k = 1, 2, \ldots, N-1 \tag{4}$$

since y_k and y_{k+1} should be distinct. This last equation states that the input region boundaries should be chosen midway between output levels. In order to simplify the other necessary condition for minimum error, the error function g(.) must be specified.

Minimum Mean Square Quantization Error

An often used criterion is the mean square error. In this case $g(a)=a^2$. A necessary condition for maximizing the signal-to-quantizing noise ratio is (from (3)),

$$\int_{x_{k-1}}^{x_k} (x-y_k)\, p(x)\, dx = 0,\ k = 1,\ 2,\ldots,N-1 \tag{5}$$

This condition indicates that y_k should be the centroid of the area of $p(x)$ for $y_{k-1} < x < x_k$. In general the two necessary conditions (4) and (5) must be solved iteratively to get a quantizer which minimizes the mean square error.

The question of the uniqueness of the quantizer which satisfies (4) and (5) has been addressed by Fleischer [3]. He gives a sufficient condition for uniqueness of the stationary point described by (4) and (5),

$$\frac{d^2}{dx^2} \log\, (p(x)) < 0.$$

It is shown in Appendix A that the Gaussian density and the Lapacian (double sided exponential) density functions satisfy this criterion and hence have unique stationary points which yield a minimum mean square error.

Iterative Solution

An iterative procedure to find a minimum mean square error quantizer might proceed as follows

i) Set $k=0$. Choose a value for y_1.

   The value of x_o is known, $x_o = -\infty$.

ii) Solve equation (5) for x_k.

iii) Find y_{k+1} from (4).

iv) Repeat steps ii) and iii) for $k=1, \ldots, N-1$.

If y_N is not the centroid of the area of $p(x)$ from x_{N-1} to ∞, the initial point y_1 has to be altered. The whole procedure is then repeated for a new initial point y_1. The procedure is terminated when the y_i have been determined to sufficient accuracy.

Numerical results have been obtained for several specific probability density functions. The first is the unit variance Gaussian density,

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The other two belong to the family of double sided gamma densities,

$$p(x) = \frac{\lambda(\lambda|x|)^{a-1} e^{-\lambda|x|}}{2\Gamma(a)}$$

where Γ (a) is the gamma function. The variance of this density function is (see Appendix A),

$$\sigma^2 = \frac{a(a+1)}{\lambda^2}$$

For a=1, p(x) is the exponential or Laplace density; for a=½ p(x) is the so called gamma density. Both of these densities have been used to approximate the measured probability density function of speech [2].

Appendix B shows how by suitable transformation all the integrals required for the iterative procedure may be obtained from the integral of the gamma density.

The resulting minimum mean square error quantizers are shown in Tables I, II, and III for the number of levels running from N=1 to N=64. Only the positive output levels y_k are tabulated, the others are obtained by symmetry. The quantizer break points be midway between the quantizer output levels (see (4)). The results for the Gaussian density agree with those given in [1]. The results for the other two densities differ somewhat from the results given in [2]. This seems primarily to be due to the approximations in the procedure used in [2].

A listing of the computer program used is given in Appendix C. Another program which uses a tabulated probability density function is given in Appendix D. For a sufficiently densely sampled probability density function, excellent agreement was obtained for the two programs.

ADDENDUM

The calculation of optimal quantizers for Laplace densities has been reported recently in the literature[1]. In addition Noel and Zelinski[2] have brought attention to a simple recursion discovered by Nitadori[3] which can be used to get the optimal quantizer for Laplace densities.

[1] W.C. Adams Jr., and C.E. Giesler, "Quantizing Characteristics for Having Laplacian Amplitude Probability Density Function", IEEE Trans. Commun., Vol. COM-26, pp. 1295-1297, Aug. 1978.

[2] P. Noel and R. Zelinski, "Comments on 'Quantizing Characteristics for Signals Having Laplacian Amplitude Probability Density Function'", IEEE Trans. Commun., Vol. COM-27, pp. 1259-1260.

[3] K. Nitadori, "Statistical Analysis of DPCM", Electron. Commun. in Japan, Vol. 48, pp. 17-26, Feb. 1965.

TABLE I - Optimal Quantizers for a Gaussian Density

	N = 1	N = 2	N = 3	N = 4	N = 5	N = 6	N = 7	N = 8
	0.0000	0.7979	0.0000 1.224	0.4528 1.510	0.0000 0.7646 1.724	0.3177 1.000 1.894	0.0000 0.5606 1.188 2.033	0.2451 0.7560 1.344 2.152
ERROR	1.000E 00	3.634E-01	1.902E-01	1.175E-01	7.994E-02	5.798E-02	4.400E-02	3.455E-02
ENTROPY	0.000	1.000	1.536	1.911	2.203	2.443	2.647	2.825

	N = 9	N = 10	N = 11	N = 12	N = 13	N = 14	N = 15	N = 16
	0.0000 0.4436 0.9188 1.476 2.255	0.1996 0.6099 1.058 1.591 2.345	0.0000 0.3675 0.7524 1.179 1.693 2.426	0.1684 0.5118 0.8768 1.286 1.783 2.498	0.0000 0.3138 0.6383 0.9869 1.381 1.865 2.565	0.1457 0.4413 0.7504 1.086 1.468 1.939 2.625	0.0000 0.2739 0.5548 0.8511 1.175 1.546 2.006 2.681	0.1284 0.3880 0.6568 0.9423 1.256 1.618 2.069 2.733
ERROR	2.785E-02	2.294E-02	1.922E-02	1.634E-02	1.406E-02	1.223E-02	1.074E-02	9.501E-03
ENTROPY	2.983	3.125	3.254	3.372	3.481	3.582	3.677	3.765

	N = 17	N = 18	N = 19	N = 20	N = 21	N = 22	N = 23	N = 24
	0.0000 0.2430 0.4909 0.7493 1.026 1.331 1.684 2.127 2.781	0.1148 0.3463 0.5843 0.8339 1.102 1.400 1.746 2.181 2.826	0.0000 0.2184 0.4404 0.6698 0.9117 1.173 1.464 1.803 2.231 2.868	0.1038 0.3128 0.5265 0.7485 0.9836 1.238 1.523 1.857 2.279 2.908	0.0000 0.1984 0.3994 0.6059 0.8214 1.051 1.300 1.579 1.907 2.323 2.946	0.0947 0.2852 0.4792 0.6795 0.8893 1.113 1.357 1.632 1.955 2.365 2.981	0.0000 0.1817 0.3654 0.5533 0.7480 0.9527 1.172 1.411 1.681 2.000 2.405 3.015	0.0871 0.2621 0.4399 0.6224 0.8121 1.012 1.227 1.462 1.728 2.042 2.443 3.047
ERROR	8.467E-03	7.593E-03	6.848E-03	6.208E-03	5.653E-03	5.170E-03	4.746E-03	4.372E-03
ENTROPY	3.849	3.928	4.003	4.074	4.141	4.206	4.268	4.327

	N = 25	N = 26	N = 27	N = 28	N = 29	N = 30	N = 31	N = 32
	0.0000	0.0806	0.0000	0.0750	0.0000	0.0702	0.0000	0.0659
	0.1676	0.2425	0.1556	0.2256	0.1451	0.2109	0.1360	0.1981
	0.3368	0.4065	0.3124	0.3779	0.2913	0.3531	0.2729	0.3314
	0.5093	0.5743	0.4718	0.5332	0.4396	0.4977	0.4115	0.4667
	0.6870	0.7476	0.6354	0.6929	0.5911	0.6459	0.5527	0.6049
	0.8722	0.9288	0.8048	0.8588	0.7474	0.7989	0.6978	0.7471
	1.068	1.121	0.9823	1.033	0.9099	0.9585	0.8479	0.8946
	1.279	1.328	1.171	1.218	1.081	1.127	1.005	1.049
	1.510	1.555	1.374	1.419	1.263	1.306	1.170	1.212
	1.772	1.814	1.599	1.640	1.461	1.501	1.347	1.386
	2.082	2.121	1.854	1.892	1.679	1.717	1.539	1.576
	2.479	2.513	2.157	2.192	1.928	1.963	1.753	1.787
	3.078	3.108	2.546	2.578	2.225	2.257	1.997	2.029
			3.136	3.163	2.608	2.637	2.288	2.318
					3.189	3.214	2.664	2.691
							3.238	3.261
ERROR	4.041E-03	3.746E-03	3.482E-03	3.246E-03	3.032E-03	2.839E-03	2.664E-03	2.505E-03
ENTROPY	4.384	4.439	4.492	4.543	4.592	4.639	4.685	4.730

	N = 33	N = 34	N = 35	N = 36	N = 37	N = 38	N = 39	N = 40
	0.0000	0.0621	0.0000	0.0587	0.0000	0.0557	0.0000	0.0530
	0.1280	0.1867	0.1208	0.1765	0.1145	0.1674	0.1087	0.1592
	0.2567	0.3122	0.2423	0.2951	0.2294	0.2798	0.2179	0.2660
	0.3868	0.4394	0.3649	0.4151	0.3454	0.3934	0.3279	0.3738
	0.5191	0.5690	0.4894	0.5371	0.4629	0.5087	0.4392	0.4831
	0.6546	0.7018	0.6165	0.6618	0.5827	0.6263	0.5524	0.5944
	0.7941	0.8390	0.7470	0.7901	0.7052	0.7468	0.6681	0.7082
	0.9391	0.9816	0.8818	0.9229	0.8315	0.8711	0.7867	0.8251
	1.091	1.131	1.022	1.061	0.9623	1.000	0.9092	0.9459
	1.252	1.290	1.170	1.207	1.099	1.135	1.036	1.071
	1.424	1.460	1.326	1.362	1.243	1.277	1.170	1.203
	1.612	1.646	1.495	1.528	1.395	1.428	1.310	1.342
	1.820	1.852	1.678	1.710	1.560	1.591	1.459	1.490
	2.060	2.089	1.883	1.912	1.740	1.769	1.621	1.650
	2.346	2.374	2.118	2.146	1.941	1.969	1.797	1.825
	2.717	2.742	2.400	2.426	2.172	2.198	1.995	2.021
	3.283	3.305	2.766	2.789	2.450	2.474	2.223	2.247
			3.326	3.346	2.811	2.833	2.497	2.520
					3.365	3.384	2.854	2.875
							3.403	3.421
ERROR	2.359E-03	2.226E-03	2.104E-03	1.991E-03	1.887E-03	1.792E-03	1.703E-03	1.621E-03
ENTROPY	4.773	4.815	4.856	4.895	4.934	4.972	5.008	5.044

	N = 41	N = 42	N = 43	N = 44	N = 45	N = 46	N = 47	N = 48
	Ø.ØØØØ	Ø.Ø5Ø5	Ø.ØØØØ	Ø.Ø483	Ø.ØØØØ	Ø.Ø462	Ø.ØØØØ	Ø.Ø443
	Ø.1Ø35	Ø.1518	Ø.Ø988	Ø.145Ø	Ø.Ø945	Ø.1388	Ø.Ø9Ø6	Ø.1331
	Ø.2Ø74	Ø.2535	Ø.1979	Ø.2422	Ø.1893	Ø.2318	Ø.1813	Ø.2223
	Ø.3121	Ø.3562	Ø.2977	Ø.34Ø1	Ø.2846	Ø.3254	Ø.2726	Ø.312Ø
	Ø.4179	Ø.46Ø1	Ø.3985	Ø.4391	Ø.38Ø8	Ø.42ØØ	Ø.3647	Ø.4Ø26
	Ø.5252	Ø.5657	Ø.5ØØ6	Ø.5396	Ø.4783	Ø.5159	Ø.4578	Ø.4943
	Ø.6347	Ø.6734	Ø.6Ø46	Ø.642Ø	Ø.5772	Ø.6134	Ø.5523	Ø.5873
	Ø.7467	Ø.7838	Ø.71Ø7	Ø.7466	Ø.6781	Ø.7129	Ø.6484	Ø.6822
	Ø.862Ø	Ø.8975	Ø.8196	Ø.8541	Ø.7813	Ø.8148	Ø.7466	Ø.7791
	Ø.9812	1.Ø15	Ø.9318	Ø.9649	Ø.8874	Ø.9196	Ø.8472	Ø.8785
	1.1Ø5	1.138	1.Ø48	1.Ø8Ø	Ø.9969	1.Ø28	Ø.95Ø7	Ø.98Ø9
	1.235	1.266	1.169	1.2ØØ	1.111	1.14Ø	1.Ø58	1.Ø87
	1.373	1.4Ø2	1.297	1.326	1.229	1.258	1.169	1.197
	1.519	1.548	1.431	1.459	1.354	1.381	1.285	1.312
	1.677	1.7Ø5	1.575	1.6Ø2	1.486	1.512	1.4Ø8	1.434
	1.851	1.877	1.731	1.756	1.628	1.653	1.538	1.563
	2.Ø46	2.Ø7Ø	1.9Ø2	1.926	1.781	1.8Ø5	1.677	1.7Ø1
	2.271	2.293	2.Ø94	2.117	1.949	1.972	1.828	1.851
	2.542	2.563	2.315	2.337	2.139	2.16Ø	1.994	2.Ø16
	2.895	2.914	2.583	2.6Ø3	2.358	2.378	2.181	2.2Ø2
	3.438	3.455	2.933	2.951	2.623	2.642	2.398	2.417
			3.472	3.488	2.969	2.986	2.66Ø	2.678
					3.5Ø4	3.519	3.ØØ3	3.Ø2Ø
							3.534	3.548
ERROR	1.544E-Ø3	1.473E-Ø3	1.4Ø7E-Ø3	1.345E-Ø3	1.287E-Ø3	1.233E-Ø3	1.182E-Ø3	1.134E-Ø3
ENTROPY	5.Ø79	5.113	5.146	5.179	5.21Ø	5.242	5.272	5.3Ø2

	N = 49	N = 5Ø	N = 51	N = 52	N = 53	N = 54	N = 55	N = 56
	Ø.ØØØØ	Ø.Ø426	Ø.ØØØØ	Ø.Ø41Ø	Ø.ØØØØ	Ø.Ø395	Ø.ØØØØ	Ø.Ø381
	Ø.Ø869	Ø.1279	Ø.Ø836	Ø.123Ø	Ø.Ø8Ø5	Ø.1186	Ø.Ø776	Ø.1144
	Ø.1741	Ø.2135	Ø.1673	Ø.2Ø54	Ø.1611	Ø.1979	Ø.1553	Ø.19Ø9
	Ø.2616	Ø.2996	Ø.2515	Ø.2882	Ø.2421	Ø.2776	Ø.2334	Ø.2678
	Ø.3499	Ø.3865	Ø.3363	Ø.3717	Ø.3236	Ø.3579	Ø.3119	Ø.3452
	Ø.4391	Ø.4743	Ø.4218	Ø.456Ø	Ø.4Ø59	Ø.439Ø	Ø.3911	Ø.4233
	Ø.5294	Ø.5634	Ø.5Ø84	Ø.5414	Ø.4891	Ø.5211	Ø.4711	Ø.5Ø23
	Ø.6213	Ø.654Ø	Ø.5963	Ø.6282	Ø.5734	Ø.6Ø44	Ø.5522	Ø.5823
	Ø.7149	Ø.7465	Ø.6858	Ø.7166	Ø.6591	Ø.689Ø	Ø.6344	Ø.6636
	Ø.81Ø6	Ø.8411	Ø.7771	Ø.8Ø69	Ø.7464	Ø.7754	Ø.7181	Ø.7464
	Ø.9Ø89	Ø.9383	Ø.87Ø7	Ø.8994	Ø.8357	Ø.8637	Ø.8Ø36	Ø.83Ø9
	1.Ø1Ø	1.Ø39	Ø.9669	Ø.9946	Ø.9273	Ø.9544	Ø.891Ø	Ø.9175
	1.115	1.142	1.Ø66	1.Ø93	1.Ø22	1.Ø48	Ø.98Ø7	1.ØØ6
	1.224	1.251	1.169	1.195	1.119	1.144	1.Ø73	1.Ø98
	1.338	1.364	1.276	1.3Ø1	1.22Ø	1.244	1.169	1.193
	1.459	1.483	1.388	1.412	1.325	1.349	1.268	1.291
	1.587	1.61Ø	1.5Ø7	1.53Ø	1.436	1.459	1.372	1.394
	1.724	1.746	1.633	1.655	1.552	1.574	1.481	1.5Ø2
	1.873	1.894	1.768	1.789	1.677	1.698	1.596	1.617
	2.Ø37	2.Ø57	1.915	1.935	1.81Ø	1.83Ø	1.718	1.738
	2.222	2.241	2.Ø77	2.Ø97	1.955	1.975	1.85Ø	1.87Ø
	2.436	2.454	2.26Ø	2.279	2.116	2.134	1.994	2.Ø12
	2.696	2.713	2.472	2.49Ø	2.297	2.314	2.152	2.17Ø
	3.Ø36	3.Ø52	2.73Ø	2.746	2.5Ø7	2.524	2.332	2.348
	3.563	3.577	3.Ø67	3.Ø82	2.762	2.778	2.54Ø	2.556
			3.59Ø	3.6Ø4	3.Ø97	3.111	2.793	2.8Ø8
					3.617	3.629	3.126	3.139
							3.642	3.654
ERROR	1.Ø89E-Ø3	1.Ø47E-Ø3	1.ØØ7E-Ø3	9.689E-Ø4	9.336E-Ø4	9.ØØØE-Ø4	8.679E-Ø4	8.376E-Ø4
ENTROPY	5.331	5.36Ø	5.388	5.415	5.442	5.469	5.495	5.52Ø

	N = 57	N = 58	N = 59	N = 60	N = 61	N = 62	N = 63	N = 64
	0.0000	0.0368	0.0000	0.0356	0.0000	0.0345	0.0000	0.0334
	0.0749	0.1105	0.0724	0.1069	0.0701	0.1035	0.0679	0.1003
	0.1500	0.1844	0.1449	0.1783	0.1403	0.1726	0.1359	0.1673
	0.2253	0.2586	0.2177	0.2501	0.2107	0.2421	0.2041	0.2346
	0.3011	0.3333	0.2909	0.3223	0.2814	0.3119	0.2726	0.3022
	0.3774	0.4086	0.3646	0.3950	0.3527	0.3822	0.3415	0.3703
	0.4545	0.4848	0.4390	0.4685	0.4245	0.4532	0.4110	0.4390
	0.5325	0.5618	0.5142	0.5428	0.4971	0.5250	0.4811	0.5083
	0.6116	0.6400	0.5903	0.6181	0.5706	0.5976	0.5521	0.5785
	0.6920	0.7195	0.6677	0.6946	0.6451	0.6713	0.6240	0.6497
	0.7739	0.8006	0.7463	0.7725	0.7208	0.7463	0.6970	0.7219
	0.8575	0.8834	0.8266	0.8519	0.7979	0.8227	0.7712	0.7955
	0.9433	0.9684	0.9087	0.9333	0.8766	0.9007	0.8469	0.8705
	1.031	1.056	0.9928	1.017	0.9572	0.9806	0.9242	0.9472
	1.122	1.146	1.079	1.103	1.040	1.063	1.003	1.026
	1.216	1.239	1.169	1.191	1.125	1.147	1.085	1.106
	1.314	1.336	1.261	1.283	1.213	1.234	1.169	1.190
	1.416	1.438	1.358	1.379	1.304	1.325	1.255	1.276
	1.524	1.544	1.458	1.479	1.399	1.419	1.345	1.365
	1.637	1.657	1.564	1.584	1.499	1.518	1.439	1.458
	1.758	1.777	1.676	1.695	1.603	1.622	1.537	1.556
	1.888	1.907	1.796	1.814	1.714	1.732	1.641	1.659
	2.030	2.048	1.925	1.943	1.832	1.850	1.750	1.768
	2.187	2.204	2.065	2.082	1.960	1.977	1.867	1.884
	2.365	2.381	2.221	2.237	2.099	2.115	1.993	2.010
	2.571	2.587	2.397	2.412	2.253	2.269	2.131	2.147
	2.823	2.837	2.602	2.616	2.428	2.442	2.284	2.299
	3.153	3.166	2.851	2.865	2.631	2.645	2.457	2.471
	3.666	3.678	3.179	3.192	2.878	2.892	2.659	2.672
			3.689	3.701	3.204	3.217	2.905	2.917
					3.712	3.723	3.229	3.240
							3.734	3.744
ERROR	8.090E-04	7.819E-04	7.561E-04	7.315E-04	7.081E-04	6.857E-04	6.645E-04	6.438E-04
ENTROPY	5.545	5.570	5.594	5.618	5.642	5.665	5.688	5.710

TABLE II - Optimal Quantizers for a Laplace Density

	N = 1	N = 2	N = 3	N = 4	N = 5	N = 6	N = 7	N = 8
	Ø.ØØØØ	Ø.7Ø71	Ø.ØØØØ	Ø.4198	Ø.ØØØØ	Ø.2998	Ø.ØØØØ	Ø.2334
			1.414	1.834	Ø.8395	1.139	Ø.5996	Ø.833Ø
					2.254	2.554	1.439	1.672
							2.853	3.Ø87
ERROR	1.ØØØE ØØ	5.ØØØE-Ø1	2.642E-Ø1	1.762E-Ø1	1.198E-Ø1	8.987E-Ø2	6.8Ø9E-Ø2	5.448E-Ø2
ENTROPY	Ø.ØØØ	1.ØØØ	1.317	1.728	1.947	2.2Ø7	2.374	2.565

	N = 9	N = 1Ø	N = 11	N = 12	N = 13	N = 14	N = 15	N = 16
	Ø.ØØØØ	Ø.1912	Ø.ØØØØ	Ø.1619	Ø.ØØØØ	Ø.14Ø4	Ø.ØØØØ	Ø.124Ø
	Ø.4668	Ø.658Ø	Ø.3823	Ø.5442	Ø.3238	Ø.4642	Ø.28Ø9	Ø.4Ø48
	1.Ø66	1.258	Ø.8491	1.Ø11	Ø.7Ø61	Ø.8466	Ø.6Ø47	Ø.7287
	1.9Ø6	2.Ø97	1.449	1.611	1.173	1.313	Ø.987Ø	1.111
	3.32Ø	3.511	2.288	2.45Ø	1.773	1.913	1.454	1.578
			3.7Ø2	3.864	2.612	2.752	2.Ø53	2.177
					4.Ø26	4.167	2.893	3.Ø17
							4.3Ø7	4.431
ERROR	4.385E-Ø2	3.655E-Ø2	3.Ø58E-Ø2	2.621E-Ø2	2.254E-Ø2	1.972E-Ø2	1.729E-Ø2	1.537E-Ø2
ENTROPY	2.7Ø1	2.852	2.966	3.Ø91	3.189	3.296	3.382	3.475

	N = 17	N = 18	N = 19	N = 2Ø	N = 21	N = 22	N = 23	N = 24
	Ø.ØØØØ	Ø.111Ø	Ø.ØØØØ	Ø.1ØØ5	Ø.ØØØØ	Ø.Ø918	Ø.ØØØØ	Ø.Ø845
	Ø.248Ø	Ø.359Ø	Ø.222Ø	Ø.3225	Ø.2ØØ9	Ø.2927	Ø.1835	Ø.268Ø
	Ø.5288	Ø.6398	Ø.47ØØ	Ø.57Ø4	Ø.4229	Ø.5147	Ø.3845	Ø.4689
	Ø.8526	Ø.9636	Ø.75Ø8	Ø.8513	Ø.67Ø9	Ø.7627	Ø.6Ø65	Ø.69Ø9
	1.235	1.346	1.Ø75	1.175	Ø.9518	1.Ø44	Ø.8544	Ø.9389
	1.7Ø2	1.813	1.457	1.557	1.276	1.367	1.135	1.22Ø
	2.3Ø1	2.412	1.924	2.Ø24	1.658	1.75Ø	1.459	1.544
	3.141	3.252	2.523	2.624	2.125	2.216	1.841	1.926
	4.555	4.666	3.363	3.463	2.724	2.816	2.3Ø8	2.393
			4.777	4.878	3.564	3.656	2.9Ø8	2.992
					4.978	5.Ø7Ø	3.747	3.832
							5.162	5.246
ERROR	1.369E-Ø2	1.232E-Ø2	1.11ØE-Ø2	1.ØØ9E-Ø2	9.187E-Ø3	8.421E-Ø3	7.727E-Ø3	7.132E-Ø3
ENTROPY	3.552	3.634	3.7Ø4	3.778	3.842	3.9Ø8	3.967	4.Ø28

	N = 25	N = 26	N = 27	N = 28	N = 29	N = 30	N = 31	N = 32
	0.0000	0.0782	0.0000	0.0729	0.0000	0.0682	0.0000	0.0640
	0.1689	0.2471	0.1565	0.2293	0.1457	0.2139	0.1363	0.2004
	0.3525	0.4307	0.3254	0.3982	0.3022	0.3703	0.2820	0.3461
	0.5534	0.6316	0.5089	0.5818	0.4711	0.5392	0.4385	0.5025
	0.7754	0.8536	0.7099	0.7827	0.6546	0.7228	0.6074	0.6714
	1.023	1.102	0.9318	1.005	0.8556	0.9237	0.7909	0.8550
	1.304	1.382	1.180	1.253	1.078	1.146	0.9919	1.056
	1.628	1.706	1.461	1.534	1.326	1.394	1.214	1.278
	2.010	2.089	1.784	1.857	1.606	1.675	1.462	1.526
	2.477	2.555	2.167	2.240	1.930	1.998	1.743	1.807
	3.077	3.155	2.634	2.706	2.313	2.381	2.067	2.131
	3.916	3.994	3.233	3.306	2.779	2.847	2.449	2.513
	5.330	5.409	4.073	4.146	3.379	3.447	2.916	2.980
			5.487	5.560	4.218	4.287	3.515	3.579
					5.633	5.701	4.355	4.419
							5.769	5.833
ERROR	6.589E-03	6.119E-03	5.685E-03	5.307E-03	4.956E-03	4.647E-03	4.358E-03	4.102E-03
ENTROPY	4.082	4.139	4.189	4.241	4.288	4.337	4.381	4.427

	N = 33	N = 34	N = 35	N = 36	N = 37	N = 38	N = 39	N = 40
	0.0000	0.0604	0.0000	0.0571	0.0000	0.0542	0.0000	0.0516
	0.1281	0.1885	0.1208	0.1779	0.1143	0.1685	0.1084	0.1600
	0.2644	0.3248	0.2489	0.3060	0.2351	0.2893	0.2227	0.2743
	0.4101	0.4705	0.3852	0.4424	0.3632	0.4174	0.3435	0.3951
	0.5666	0.6270	0.5309	0.5881	0.4995	0.5537	0.4716	0.5232
	0.7355	0.7959	0.6874	0.7445	0.6452	0.6994	0.6080	0.6595
	0.9190	0.9794	0.8563	0.9134	0.8017	0.8559	0.7537	0.8052
	1.120	1.180	1.040	1.097	0.9706	1.025	0.9101	0.9617
	1.342	1.402	1.241	1.298	1.154	1.208	1.079	1.131
	1.590	1.650	1.463	1.520	1.355	1.409	1.263	1.314
	1.871	1.931	1.711	1.768	1.577	1.631	1.464	1.515
	2.195	2.255	1.992	2.049	1.825	1.879	1.686	1.737
	2.577	2.637	2.315	2.373	2.106	2.160	1.933	1.985
	3.044	3.104	2.698	2.755	2.430	2.484	2.214	2.266
	3.643	3.704	3.165	3.222	2.812	2.866	2.538	2.590
	4.483	4.543	3.764	3.821	3.279	3.333	2.920	2.972
	5.897	5.957	4.604	4.661	3.878	3.933	3.387	3.439
			6.018	6.075	4.718	4.772	3.987	4.038
					6.132	6.186	4.826	4.878
							6.241	6.292
ERROR	3.862E-03	3.647E-03	3.446E-03	3.264E-03	3.095E-03	2.939E-03	2.793E-03	2.661E-03
ENTROPY	4.468	4.511	4.551	4.591	4.629	4.667	4.703	4.739

	N = 41	N = 42	N = 43	N = 44	N = 45	N = 46	N = 47	N = 48
	Ø.ØØØØ	Ø.Ø492	Ø.ØØØØ	Ø.Ø47Ø	Ø.ØØØØ	Ø.Ø45Ø	Ø.ØØØØ	Ø.Ø432
	Ø.1Ø32	Ø.1524	Ø.Ø984	Ø.1454	Ø.Ø94Ø	Ø.139Ø	Ø.Ø9ØØ	Ø.1332
	Ø.2116	Ø.26Ø8	Ø.2Ø16	Ø.2486	Ø.1924	Ø.2374	Ø.1841	Ø.2272
	Ø.3259	Ø.3751	Ø.31ØØ	Ø.357Ø	Ø.2956	Ø.34Ø6	Ø.2824	Ø.3256
	Ø.4467	Ø.4959	Ø.4243	Ø.4713	Ø.4Ø4Ø	Ø.449Ø	Ø.3856	Ø.4288
	Ø.5748	Ø.624Ø	Ø.5451	Ø.5921	Ø.5183	Ø.5633	Ø.4941	Ø.5372
	Ø.7111	Ø.76Ø3	Ø.6732	Ø.72Ø2	Ø.6391	Ø.6841	Ø.6Ø83	Ø.6515
	Ø.8568	Ø.9Ø6Ø	Ø.8Ø95	Ø.8565	Ø.7672	Ø.8122	Ø.7291	Ø.7723
	1.Ø13	1.Ø62	Ø.9552	1.ØØ2	Ø.9Ø35	Ø.9486	Ø.8572	Ø.9ØØ4
	1.182	1.231	1.112	1.159	1.Ø49	1.Ø94	Ø.9936	1.Ø37
	1.366	1.415	1.281	1.328	1.2Ø6	1.251	1.139	1.182
	1.567	1.616	1.464	1.511	1.375	1.42Ø	1.296	1.339
	1.789	1.838	1.665	1.712	1.558	1.6Ø3	1.465	1.5Ø8
	2.Ø37	2.Ø86	1.887	1.934	1.759	1.8Ø4	1.648	1.691
	2.317	2.367	2.135	2.182	1.981	2.Ø26	1.849	1.892
	2.641	2.691	2.416	2.463	2.229	2.274	2.Ø71	2.114
	3.Ø24	3.Ø73	2.74Ø	2.787	2.51Ø	2.555	2.319	2.362
	3.49Ø	3.54Ø	3.122	3.169	2.834	2.879	2.6ØØ	2.643
	4.Ø9Ø	4.139	3.589	3.636	3.216	3.261	2.924	2.967
	4.93Ø	4.979	4.188	4.235	3.683	3.728	3.3Ø6	3.349
	6.344	6.393	5.Ø28	5.Ø75	4.282	4.327	3.773	3.816
			6.442	6.489	5.122	5.167	4.372	4.416
					6.536	6.581	5.212	5.255
							6.626	6.669
ERROR	2.533E-Ø3	2.42ØE-Ø3	2.3Ø9E-Ø3	2.21ØE-Ø3	2.114E-Ø3	2.Ø27E-Ø3	1.941E-Ø3	1.864E-Ø3
ENTROPY	4.773	4.8Ø7	4.84Ø	4.873	4.9Ø4	4.935	4.965	4.995

	N = 49	N = 50	N = 51	N = 52	N = 53	N = 54	N = 55	N = 56
	0.0000	0.0415	0.0000	0.0399	0.0000	0.0385	0.0000	0.0371
	0.0864	0.1279	0.0830	0.1229	0.0799	0.1183	0.0770	0.1141
	0.1764	0.2179	0.1694	0.2093	0.1628	0.2013	0.1568	0.1940
	0.2704	0.3119	0.2594	0.2993	0.2492	0.2877	0.2398	0.2769
	0.3688	0.4103	0.3534	0.3933	0.3392	0.3777	0.3262	0.3633
	0.4720	0.5135	0.4518	0.4917	0.4333	0.4717	0.4162	0.4533
	0.5804	0.6219	0.5550	0.5949	0.5316	0.5701	0.5102	0.5474
	0.6947	0.7362	0.6634	0.7033	0.6348	0.6733	0.6086	0.6457
	0.8155	0.8570	0.7777	0.8176	0.7433	0.7817	0.7118	0.7489
	0.9436	0.9851	0.8985	0.9384	0.8576	0.8960	0.8202	0.8574
	1.080	1.121	1.027	1.067	0.9784	1.017	0.9345	0.9716
	1.226	1.267	1.163	1.203	1.106	1.145	1.055	1.092
	1.382	1.424	1.309	1.349	1.243	1.281	1.183	1.221
	1.551	1.592	1.465	1.505	1.388	1.427	1.320	1.357
	1.735	1.776	1.634	1.674	1.545	1.583	1.465	1.503
	1.935	1.977	1.818	1.857	1.714	1.752	1.622	1.659
	2.157	2.199	2.018	2.058	1.897	1.936	1.791	1.828
	2.405	2.447	2.240	2.280	2.098	2.137	1.974	2.011
	2.686	2.728	2.488	2.528	2.320	2.359	2.175	2.212
	3.010	3.052	2.769	2.809	2.568	2.607	2.397	2.434
	3.392	3.434	3.093	3.133	2.849	2.888	2.645	2.682
	3.859	3.901	3.475	3.515	3.173	3.211	2.926	2.963
	4.459	4.500	3.942	3.982	3.555	3.594	3.250	3.287
	5.298	5.340	4.542	4.582	4.022	4.061	3.632	3.669
	6.713	6.754	5.381	5.421	4.622	4.660	4.099	4.136
			6.796	6.835	5.461	5.500	4.699	4.736
					6.875	6.914	5.538	5.575
							6.952	6.989
ERROR	1.790E-03	1.721E-03	1.655E-03	1.593E-03	1.535E-03	1.480E-03	1.427E-03	1.379E-03
ENTROPY	5.024	5.053	5.080	5.108	5.135	5.162	5.187	5.213

	N = 57	N = 58	N = 59	N = 60	N = 61	N = 62	N = 63	N = 64
	0.0000	0.0359	0.0000	0.0347	0.0000	0.0336	0.0000	0.0326
	0.0743	0.1101	0.0718	0.1065	0.0694	0.1030	0.0672	0.0998
	0.1512	0.1871	0.1460	0.1807	0.1412	0.1748	0.1366	0.1692
	0.2311	0.2670	0.2230	0.2577	0.2154	0.2490	0.2084	0.2409
	0.3141	0.3500	0.3028	0.3376	0.2924	0.3260	0.2826	0.3152
	0.4004	0.4363	0.3858	0.4205	0.3723	0.4059	0.3596	0.3922
	0.4905	0.5264	0.4722	0.5069	0.4552	0.4888	0.4395	0.4720
	0.5845	0.6204	0.5622	0.5969	0.5416	0.5752	0.5225	0.5550
	0.6829	0.7188	0.6563	0.6910	0.6316	0.6652	0.6088	0.6414
	0.7860	0.8219	0.7546	0.7893	0.7257	0.7593	0.6988	0.7314
	0.8945	0.9304	0.8578	0.8925	0.8240	0.8576	0.7929	0.8254
	1.009	1.045	0.9663	1.001	0.9272	0.9608	0.8913	0.9238
	1.130	1.165	1.081	1.115	1.036	1.069	0.9944	1.027
	1.258	1.294	1.201	1.236	1.150	1.184	1.103	1.135
	1.394	1.430	1.329	1.364	1.271	1.304	1.217	1.250
	1.540	1.576	1.466	1.500	1.399	1.432	1.338	1.371
	1.696	1.732	1.611	1.646	1.535	1.569	1.466	1.499
	1.865	1.901	1.768	1.803	1.681	1.714	1.602	1.635
	2.049	2.084	1.937	1.972	1.837	1.871	1.748	1.781
	2.250	2.285	2.120	2.155	2.006	2.040	1.905	1.937
	2.472	2.507	2.321	2.356	2.190	2.223	2.073	2.106
	2.720	2.755	2.543	2.578	2.391	2.424	2.257	2.290
	3.000	3.036	2.791	2.826	2.613	2.646	2.458	2.491
	3.324	3.360	3.072	3.107	2.861	2.894	2.680	2.712
	3.707	3.742	3.396	3.431	3.142	3.175	2.928	2.960
	4.173	4.209	3.778	3.813	3.465	3.499	3.209	3.241
	4.773	4.809	4.245	4.280	3.848	3.881	3.533	3.565
	5.612	5.648	4.845	4.879	4.314	4.348	3.915	3.947
	7.027	7.062	5.684	5.719	4.914	4.948	4.382	4.414
			7.098	7.133	5.754	5.787	4.981	5.014
					7.168	7.201	5.821	5.853
							7.235	7.268
ERROR	1.331E-03	1.287E-03	1.244E-03	1.204E-03	1.165E-03	1.129E-03	1.094E-03	1.060E-03
ENTROPY	5.238	5.263	5.286	5.311	5.334	5.357	5.379	5.402

TABLE III - Optimal Quantizers for a Gamma Density

	N = 1	N = 2	N = 3	N = 4	N = 5	N = 6	N = 7	N = 8
	0.0000	0.5774	0.0000	0.3132	0.0000	0.2097	0.0000	0.1554
			1.851	2.223	1.039	1.291	0.7101	0.8994
					3.034	3.307	1.853	2.057
							3.905	4.121
ERROR	1.000E 00	6.667E-01	2.961E-01	2.318E-01	1.395E-01	1.171E-01	8.075E-02	7.047E-02
ENTROPY	0.000	1.000	0.938	1.580	1.501	1.991	1.904	2.309

	N = 9	N = 10	N = 11	N = 12	N = 13	N = 14	N = 15	N = 16
	0.0000	0.1223	0.0000	0.1001	0.0000	0.0844	0.0000	0.0726
	0.5336	0.6837	0.4241	0.5478	0.3500	0.4547	0.2967	0.3871
	1.325	1.488	1.025	1.159	0.8312	0.9453	0.6962	0.7949
	2.510	2.682	1.852	1.993	1.462	1.582	1.203	1.307
	4.595	4.773	3.061	3.207	2.308	2.433	1.851	1.959
			5.166	5.317	3.533	3.662	2.711	2.822
					5.653	5.785	3.947	4.061
							6.078	6.195
ERROR	5.258E-02	4.704E-02	3.693E-02	3.362E-02	2.736E-02	2.521E-02	2.107E-02	1.961E-02
ENTROPY	2.218	2.567	2.476	2.785	2.695	2.973	2.885	3.139

	N = 17	N = 18	N = 19	N = 20	N = 21	N = 22	N = 23	N = 24
	0.0000	0.0636	0.0000	0.0564	0.0000	0.0506	0.0000	0.0458
	0.2566	0.3361	0.2255	0.2961	0.2007	0.2642	0.1804	0.2380
	0.5969	0.6837	0.5210	0.5983	0.4612	0.5308	0.4130	0.4762
	1.019	1.111	0.8817	0.9633	0.7752	0.8487	0.6904	0.7572
	1.541	1.636	1.318	1.402	1.148	1.225	1.016	1.085
	2.201	2.299	1.850	1.938	1.594	1.672	1.398	1.469
	3.070	3.171	2.518	2.608	2.134	2.215	1.850	1.923
	4.315	4.418	3.395	3.486	2.808	2.891	2.396	2.471
	6.455	6.560	4.647	4.740	3.691	3.775	3.075	3.151
			6.794	6.889	4.948	5.034	3.963	4.040
					7.101	7.188	5.225	5.303
							7.383	7.463
ERROR	1.673E-02	1.569E-02	1.360E-02	1.283E-02	1.128E-02	1.069E-02	9.496E-03	9.044E-03
ENTROPY	3.052	3.288	3.203	3.422	3.339	3.545	3.463	3.657

	N = 25	N = 26	N = 27	N = 28	N = 29	N = 30	N = 31	N = 32
	0.0000	0.0417	0.0000	0.0383	0.0000	0.0354	0.0000	0.0328
	0.1637	0.2163	0.1496	0.1979	0.1375	0.1823	0.1272	0.1687
	0.3733	0.4311	0.3402	0.3933	0.3121	0.3612	0.2879	0.3337
	0.6214	0.6825	0.5642	0.6204	0.5160	0.5681	0.4749	0.5234
	0.9093	0.9729	0.8220	0.8805	0.7491	0.8033	0.6874	0.7378
	1.243	1.308	1.117	1.178	1.014	1.070	0.9268	0.9789
	1.631	1.698	1.456	1.518	1.314	1.372	1.197	1.250
	2.088	2.157	1.849	1.913	1.658	1.717	1.502	1.556
	2.639	2.709	2.311	2.376	2.055	2.115	1.849	1.905
	3.323	3.394	2.866	2.931	2.521	2.581	2.249	2.306
	4.214	4.286	3.553	3.619	3.078	3.140	2.718	2.775
	5.480	5.553	4.448	4.515	3.768	3.831	3.278	3.336
	7.643	7.717	5.717	5.785	4.666	4.729	3.971	4.029
			7.884	7.953	5.938	6.002	4.871	4.930
					8.108	8.173	6.146	6.206
							8.319	8.380
ERROR	8.110E-03	7.751E-03	7.007E-03	6.716E-03	6.112E-03	5.876E-03	5.378E-03	5.184E-03
ENTROPY	3.577	3.762	3.683	3.859	3.782	3.950	3.875	4.036

	N = 33	N = 34	N = 35	N = 36	N = 37	N = 38	N = 39	N = 40
	0.0000	0.0306	0.0000	0.0286	0.0000	0.0269	0.0000	0.0253
	0.1182	0.1569	0.1103	0.1466	0.1033	0.1374	0.0971	0.1292
	0.2671	0.3097	0.2488	0.2888	0.2327	0.2703	0.2184	0.2539
	0.4395	0.4848	0.4087	0.4511	0.3816	0.4215	0.3577	0.3953
	0.6345	0.6816	0.5886	0.6329	0.5486	0.5902	0.5133	0.5526
	0.8529	0.9016	0.7893	0.8349	0.7339	0.7769	0.6854	0.7260
	1.097	1.147	1.012	1.059	0.9389	0.9830	0.8749	0.9165
	1.371	1.422	1.260	1.308	1.166	1.211	1.083	1.126
	1.680	1.731	1.538	1.586	1.417	1.463	1.313	1.356
	2.030	2.083	1.849	1.898	1.697	1.743	1.567	1.611
	2.433	2.486	2.202	2.252	2.010	2.058	1.849	1.893
	2.904	2.958	2.607	2.658	2.366	2.413	2.165	2.210
	3.467	3.521	3.080	3.132	2.773	2.821	2.522	2.567
	4.162	4.217	3.645	3.697	3.248	3.297	2.931	2.977
	5.064	5.120	4.342	4.395	3.815	3.864	3.407	3.454
	6.341	6.398	5.247	5.300	4.513	4.563	3.976	4.023
	8.517	8.575	6.526	6.580	5.420	5.470	4.676	4.724
			8.704	8.759	6.701	6.752	5.584	5.632
					8.881	8.933	6.867	6.916
							9.050	9.099
ERROR	4.770E-03	4.609E-03	4.260E-03	4.122E-03	3.827E-03	3.709E-03	3.457E-03	3.355E-03
ENTROPY	3.962	4.116	4.044	4.192	4.121	4.265	4.195	4.334

	N = 41	N = 42	N = 43	N = 44	N = 45	N = 46	N = 47	N = 48
	0.0000	0.0239	0.0000	0.0226	0.0000	0.0215	0.0000	0.0204
	0.0915	0.1219	0.0865	0.1153	0.0820	0.1093	0.0779	0.1039
	0.2057	0.2392	0.1943	0.2261	0.1839	0.2142	0.1746	0.2034
	0.3364	0.3720	0.3173	0.3511	0.3001	0.3323	0.2846	0.3152
	0.4820	0.5192	0.4541	0.4894	0.4290	0.4625	0.4064	0.4383
	0.6425	0.6809	0.6044	0.6408	0.5702	0.6049	0.5395	0.5725
	0.8185	0.8579	0.7686	0.8060	0.7241	0.7596	0.6841	0.7180
	1.011	1.051	0.9477	0.9859	0.8913	0.9276	0.8408	0.8755
	1.222	1.263	1.143	1.182	1.073	1.110	1.010	1.046
	1.454	1.496	1.357	1.396	1.271	1.308	1.194	1.230
	1.710	1.753	1.591	1.631	1.486	1.524	1.394	1.430
	1.994	2.037	1.849	1.889	1.722	1.761	1.611	1.648
	2.312	2.355	2.134	2.175	1.981	2.020	1.848	1.886
	2.671	2.715	2.454	2.495	2.269	2.308	2.109	2.147
	3.082	3.126	2.814	2.856	2.589	2.629	2.398	2.436
	3.560	3.604	3.226	3.268	2.951	2.991	2.720	2.758
	4.130	4.175	3.705	3.748	3.364	3.405	3.083	3.121
	4.831	4.877	4.277	4.320	3.845	3.886	3.497	3.536
	5.741	5.787	4.980	5.023	4.417	4.459	3.979	4.018
	7.025	7.072	5.890	5.934	5.122	5.163	4.552	4.592
	9.210	9.257	7.176	7.221	6.034	6.076	5.258	5.298
			9.363	9.408	7.321	7.363	6.171	6.211
					9.509	9.552	7.460	7.500
							9.649	9.690
ERROR	3.137E-03	3.050E-03	2.860E-03	2.784E-03	2.617E-03	2.553E-03	2.407E-03	2.347E-03
ENTROPY	4.265	4.399	4.332	4.462	4.395	4.522	4.456	4.580

	N = 57	N = 58	N = 59	N = 6Ø	N = 61	N = 62	N = 63	N = 64
	Ø.ØØØØ	Ø.Ø163	Ø.ØØØØ	Ø.Ø157	Ø.ØØØØ	Ø.Ø151	Ø.ØØØØ	Ø.Ø146
	Ø.Ø62Ø	Ø.Ø829	Ø.Ø595	Ø.Ø796	Ø.Ø572	Ø.Ø765	Ø.Ø55Ø	Ø.Ø737
	Ø.1385	Ø.1616	Ø.1329	Ø.1551	Ø.1277	Ø.1491	Ø.1229	Ø.1435
	Ø.225Ø	Ø.2496	Ø.2158	Ø.2394	Ø.2Ø72	Ø.23ØØ	Ø.1993	Ø.2212
	Ø.32ØØ	Ø.3457	Ø.3Ø66	Ø.3314	Ø.2943	Ø.3181	Ø.2829	Ø.3Ø58
	Ø.4228	Ø.4495	Ø.4Ø5Ø	Ø.43Ø6	Ø.3884	Ø.4131	Ø.3731	Ø.3969
	Ø.5335	Ø.56Ø9	Ø.51Ø6	Ø.5369	Ø.4894	Ø.5147	Ø.4697	Ø.4942
	Ø.6521	Ø.68ØØ	Ø.6235	Ø.65Ø4	Ø.5972	Ø.6231	Ø.5728	Ø.5978
	Ø.7788	Ø.8Ø73	Ø.744Ø	Ø.7714	Ø.712Ø	Ø.7384	Ø.6824	Ø.7Ø79
	Ø.9141	Ø.9432	Ø.8724	Ø.9ØØ3	Ø.8341	Ø.861Ø	Ø.7989	Ø.8248
	1.Ø59	1.Ø88	1.ØØ9	1.Ø38	Ø.964Ø	Ø.9913	Ø.9224	Ø.9488
	1.213	1.243	1.155	1.184	1.1Ø2	1.13Ø	1.Ø54	1.Ø8Ø
	1.379	1.4Ø9	1.311	1.34Ø	1.249	1.277	1.193	1.22Ø
	1.556	1.587	1.478	1.5Ø7	1.4Ø6	1.435	1.341	1.369
	1.747	1.778	1.656	1.686	1.574	1.6Ø2	1.499	1.527
	1.953	1.984	1.848	1.878	1.753	1.782	1.668	1.695
	2.176	2.2Ø8	2.Ø55	2.Ø85	1.946	1.975	1.848	1.876
	2.419	2.451	2.279	2.3Ø9	2.154	2.183	2.Ø41	2.Ø7Ø
	2.685	2.717	2.523	2.554	2.379	2.4Ø8	2.25Ø	2.279
	2.979	3.Ø11	2.79Ø	2.821	2.623	2.653	2.476	2.5Ø4
	3.3Ø6	3.338	3.Ø84	3.115	2.891	2.921	2.721	2.75Ø
	3.673	3.7Ø6	3.412	3.443	3.186	3.216	2.989	3.Ø18
	4.Ø92	4.125	3.78Ø	3.811	3.514	3.545	3.285	3.314
	4.578	4.611	4.199	4.231	3.883	3.914	3.614	3.643
	5.156	5.189	4.686	4.718	4.3Ø3	4.334	3.983	4.Ø13
	5.866	5.899	5.265	5.297	4.791	4.822	4.4Ø4	4.434
	6.784	6.817	5.975	6.ØØ8	5.37Ø	5.4Ø1	4.892	4.922
	8.Ø77	8.111	6.894	6.926	6.Ø81	6.112	5.472	5.5Ø2
	1Ø.27	1Ø.31	8.188	8.221	7.ØØØ	7.Ø32	6.184	6.214
			1Ø.39	1Ø.42	8.296	8.328	7.1Ø4	7.134
					1Ø.49	1Ø.53	8.4ØØ	8.431
							1Ø.6Ø	1Ø.63
ERROR	1.652E-Ø3	1.617E-Ø3	1.545E-Ø3	1.514E-Ø3	1.445E-Ø3	1.419E-Ø3	1.356E-Ø3	1.331E-Ø3
ENTROPY	4.728	4.837	4.777	4.884	4.824	4.929	4.87Ø	4.972

Appendix A

Uniqueness of the Stationary Point

i) Gaussian density (unit variance)

Direct differentiation yields the following

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\frac{d^2}{dx^2} \log (p(x)) = -1$$

ii) Double Sided Gamma density

$$p(x,a) = \frac{1}{2\,\Gamma(a)} |x|^{a-1} e^{-|x|}$$

$$\frac{d^2}{dx^2} \log (p(x,a)) = \frac{-2(a-1)}{x^2}, \quad x \neq 0.$$

For a > 1, a unique stationary point exists. For a=1, a limiting argument shows that a unique stationary point exists.

Appendix B

The iterative procedure needs integrals of the form

$$\int_{-\infty}^{b} p(x)dx,$$

$$\int_{-\infty}^{b} p(x)dx, \text{ and}$$

$$\int_{-\infty}^{b} x^2 p(x)dx$$

If $p(x,a,\lambda) = \dfrac{\lambda(\lambda|x|)^{a-1}e^{-\lambda|x|}}{2\Gamma(a)}$, the generalized two-sided gamma density,

$$\int_{-\infty}^{b} p(x,a,\lambda)\ dx = \lambda \int_{-\infty}^{\lambda b} p(x,a,1)dx$$

$$\int_{-\infty}^{b} xp(x,a,\lambda)dx = \frac{a}{\lambda} \int_{-\infty}^{\lambda b} p(x,a+1,1)dx$$

$$\int_{-\infty}^{b} x^2 p(x,a,\lambda)dx = \frac{a(a+1)}{\lambda^2} \int_{-\infty}^{\lambda b} p(x,a+2,1)dx \qquad \text{(A-1)}$$

Since $p(x,a,\lambda)$ is a probability density function integrating to unity, the variance of the density function is from (A-1)

$$\sigma^2 = \frac{a(a+1)}{\lambda^2}$$

For the Gaussian density,

$$\int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}\,dx = \int_{-\infty}^{\operatorname{sgn}(b)\frac{b^2}{2\sigma^2}} p(x,\tfrac{1}{2},1)\,dx$$

$$\int_{-\infty}^{b} \frac{x}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}\,dx = \sqrt{\frac{2}{\pi}}\,\sigma \int_{-\infty}^{\operatorname{sgn}(b)\frac{b^2}{2\sigma^2}} p(x,1,1)\,dx$$

$$\int_{-\infty}^{b} \frac{x^2}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}\,dx = \sigma^2 \int_{-\infty}^{\operatorname{sgn}(b)\frac{b^2}{2\sigma^2}} p(x,3/2,1)\,dx.$$

Appendix C

Program Listing

This Appendix includes the listing for a routine to calculate the minimum mean square error quantizer for signals with Gaussian, Lapace and Gamma probability densities.

```
      SUBROUTINE QNTDSG( YQ,NLEV,IQTYPE,PARM,RMSE,ENTRPY)
C                                                  2Ø/Ø6/78 P. KABAL
C THIS SUBROUTINE DESIGNS QUANTIZERS. IT USES SUBROUTINES OPTQNT,
C QNTLEV, QNTMSE, QNTENT AND FUNCTIONS FGMØ ( ENTRIES FGM1, FGM2,
C FGSØ, FGS1, AND FGS2), GAMCDF, AND ALGAMA.
C
C YQ     - OUTPUT ARRAY OF NLEV QUANTIZER OUTPUT VALUES. THE QUANTIZER
C          BREAK POINTS OCCUR MIDWAY BETWEEN THE QUANTIZER OUTPUT
C          LEVELS. THE QUANTIZER OUTPUT LEVELS ARE IN ASCENDING ORDER.
C NLEV   - NUMBER OF QUANTIZER LEVELS
C IQTYPE - CHARACTER DESIGNATING THE TYPE OF QUANTIZER DESIRED.
C          'A' - A-LAW QUANTIZER
C          'M' - MU-LAW QUANTIZER
C          'U' - UNIFORM QUANTIZER
C          'N' - MINIMUM MEAN SQUARE ERROR QUANTIZER FOR A ZERO MEAN
C                GAUSSIAN (NORMAL) PROBABILITY DENSITY FUNCTION
C          'L' - MINIMUM MEAN SQUARE ERROR QUANTIZER FOR A ZERO MEAN
C                LAPLACIAN PROBABILITY DENSITY FUNCTION
C          'G' - MINIMUM MEAN SQUARE ERROR QUANTIZER FOR A ZERO MEAN
C                GAMMA PROBABILITY DENSITY FUNCTION
C PARM   - INPUT PARAMETER. THIS PARAMETER HAS DIFFERENT MEANINGS
C          FOR THE DIFFERENT VALUES OF IQTYPE.
C          'A' - PARAMETER A OF THE A-LAW QUANTIZER. THE QUANTIZER
C                INPUT IS ASSUMED TO LIE BETWEEN -1 AND +1.
C          'M' - PARAMETER MU OF THE MU-LAW QUANTIZER. THE QUANTIZER
C                INPUT IS ASSUMED TO LIE BETWEEN -1 AND +1.
C          'U' - THE MAXIMUM VALUE OF THE QUANTIZER INPUT (ASSUMED
C                SYMMETRIC)
C          'N' - STANDARD DEVIATION OF THE GAUSSIAN DENSITY FUNCTION
C          'L' - STANDARD DEVIATION OF THE LAPLACIAN DENSITY FUNCTION
C          'G' - STANDARD DEVIATION OF THE GAMMA DENSITY FUNCTION
C RMSE   - RESULTING MEAN SQUARE ERROR. THIS PARAMETER IS ONLY USED
C          FOR THE MINIMUM MEAN SQUARE ERROR QUANTIZERS (IQTYPE
C          EQUAL TO 'U', 'N', 'L', OR 'G').
C ENTRPY - RESULTING ENTROPY. THIS PARAMETER IS ONLY USED FOR THE
C          MINIMUM MEAN SQUARE ERROR QUANTIZERS (IQTYPE EQUAL TO 'U',
C          'N', 'L', OR 'G').
C
      BYTE IQTYPE,ALAW,MULAW,UNIF,GAUSS,LAPLC,GAMMA
      DIMENSION YQ(NLEV)
      COMMON /FCN/ S,P,AL
      EXTERNAL FGMØ,FGM1,FGM2,FGSØ,FGS1,FGS2
      DATA ALAW,MULAW,UNIF/'A','M','U'/,GAUSS,LAPLC,GAMMA/'N','L','G'/
      DATA ISYM/1/,XMN/Ø.Ø/,C1/1.442695Ø41/
C
      IF( IQTYPE.EQ.GAUSS .OR. IQTYPE.EQ.LAPLC .OR. IQTYPE.EQ.GAMMA)
     *    GO TO 5ØØ
C
C A-LAW, MU-LAW OR UNIFORM QUANTIZER
      RMSE=Ø.Ø
      ENTRPY=Ø.Ø
      XMAX=1.Ø
      IF( IQTYPE.EQ.UNIF) XMAX=PARM
      FN=NLEV
      DX=2.Ø*XMAX/FN
      XOFFS=XMAX+Ø.5*DX
C
C GENERATE A UNIFORM QUANTIZER
      DO 1ØØ I=1,NLEV
        YQ( I)=FLOAT( I)*DX-XOFFS
1ØØ   CONTINUE
      IF( IQTYPE.NE.UNIF) GO TO 15Ø
      RMSE=DX*DX/12.Ø
      ENTRPY=C1*ALOG( FN)
C
C MU-LAW QUANTIZER
15Ø   IF( IQTYPE.NE.MULAW) GO TO 3ØØ
      CONS=ALOG( 1.Ø+PARM)
      DO 2ØØ I=1,NLEV
        YQI=YQ( I)
        YQQ=( EXP( CONS*ABS( YQI))-1.Ø)/PARM
        IF( YQI.LT.Ø.Ø) YQQ=-YQQ
        YQ( I)=YQQ
2ØØ   CONTINUE
      GO TO 8ØØ
C
C A-LAW QUANTIZER
3ØØ   IF( IQTYPE.NE.ALAW) GO TO 8ØØ
      CONS=1.Ø+ALOG( PARM)
      DO 4ØØ I=1,NLEV
        YQI=YQ( I)
        CYQI=CONS*ABS( YQI)
        IF( CYQI.LE.1.Ø) YQQ=CYQI
        IF( CYQI.GT.1.Ø) YQQ=EXP( CYQI-1.Ø)
        IF( YQI.LT.Ø.Ø) YQQ=-YQQ
        YQ( I)=YQQ/PARM
4ØØ   CONTINUE
      GO TO 3ØØ
C
C MINIMUM MEAN SQUARE ERROR QUANTIZERS
5ØØ   S=PARM
      XMX=1ØØØ.*S
C
C GAMMA AND LAPLACIAN QUANTIZERS
      IF( IQTYPE.EQ.GAUSS) GO TO 6ØØ
      P=1.Ø
      IF( IQTYPE.EQ.GAMMA) P=Ø.5
      AL=SQRT( P*( P+1.Ø))/S
      CALL OPTQNT( FGMØ,FGM1,FGM2,ISYM,XMN,XMX,YQ,NLEV)
      CALL QNTMSE( FGMØ,FGM1,FGM2,XMX,YQ,NLEV,RMSE)
      CALL QNTENT( FGMØ,XMX,YQ,NLEV,ENTRPY)
      GO TO 8ØØ
C
C GAUSSIAN QUANTIZER
6ØØ   CALL OPTQNT( FGSØ,FGS1,FGS2,ISYM,XMN,XMX,YQ,NLEV)
      CALL QNTMSE( FGSØ,FGS1,FGS2,XMX,YQ,NLEV,RMSE)
      CALL QNTENT( FGSØ,XMX,YQ,NLEV,ENTRPY)
C
8ØØ   RETURN
      END
```

```
      SUBROUTINE OPTQNT(FØ,F1,F2,ISYM,XMN,XMX,YQ,NLEV)
C                                                   Ø9/Ø6/78 P. KABAL
C THIS SUBROUTINE ITERATES TO FIND THE QUANTIZER WHICH MINIMIZES THE
C MEAN SQUARE ERROR FOR A GIVEN PROBABILITY DENSITY FUNCTION. IT
C CALLS SUBROUTINE QNTLEV AND FUNCTIONS FØ, F1, F2.
C
C FØ     - EXTERNAL FUNCTION TO CALCULATE THE CUMULATIVE PROBABILITY
C          DENSITY FUNCTION, I.E. THE INTEGRAL OF P(X) FROM MINUS
C          INFINITY TO Y, WHERE P(X) IS THE PROBABILITY DENSITY
C          FUNCTION
C F1     - EXTERNAL FUNCTION TO CALCULATE THE INTEGRAL OF X*P(X)
C          FROM MINUS INFINITY TO Y, WHERE P(X) IS THE PROBABILITY
C          DENSITY FUNCTION
C F2     - EXTERNAL FUNCTION TO CALCULATE THE INTEGRAL OF X*X*P(X)
C          FROM MINUS INFINITY TO Y, WHERE P(X) IS THE PROBABILITY
C          DENSITY FUNCTION
C ISYM   - INPUT CONTROL VARIABLE
C          ISYM NONZERØ - THE PROBABILITY DENSITY FUNCTION IS
C                    SYMMETRIC ABOUT ITS MEAN XMN
C          ISYM ZERO - THE PROBABILITY DENSITY IS NOT NECESSARILY
C                    SYMMETRIC
C XMN    - ISYM ZERO - INPUT X VALUE BELOW WHICH THE PROBABILITY DENSITY
C                      MAY BE CONSIDERED TO BE ZERO
C          ISYM NONZERO - MEAN OF THE PROBABILITY DENSITY FUNCTION
C XMX    - X VALUE BEYOND WHICH THE PROBABILITY DENSITY FUNCTION
C          MAY BE CONSIDERED TO BE ZERO
C YQ     - OUTPUT ARRAY OF NLEV QUANTIZER OUTPUT VALUES. THE QUANTIZER
C          BREAK POINTS OCCUR MIDWAY BETWEEN THE QUANTIZER OUTPUT
C          LEVELS. THE QUANTIZER OUTPUT LEVELS ARE IN ASCENDING ORDER.
C NLEV   - NUMBER OF QUANTIZER LEVELS
C
      DIMENSION YQ(NLEV)
      LOGICAL STRDL,SYODD
      EXTERNAL FØ,F1,F2
      DATA NIMX/1ØØ/
C
C FIND THE STANDARD DEVIATION OF THE DISTRIBUTION
      AM=XMN
      IF(ISYM.EQ.Ø) AM=F1(XMX)
      S=SQRT(F2(XMX)-AM*AM)
C
C INITIALIZATION
      YQ(1)=AM
      IF(NLEV.LE.1) GO TO 8ØØ
      YQØ=XMN
      STRDL=.FALSE.
      DYQ=Ø.5*S
      EPS=5E-7*S
      XST=YQØ
      NL=NLEV
      IF(ISYM.NE.Ø) NL=NL/2
      SYODD=ISYM.NE.Ø .AND. 2*NL.NE.NLEV
C
C ITERATE TO FIND A MINIMUM MEAN SQUARE ERROR QUANTIZER
      DO 4ØØ ITER=1,NIMX
        YQ(1)=YQØ+DYQ
        IF(SYODD) XST=Ø.5*(AM+YQ(1))
        CALL QNTLEV(FØ,F1,XST,XMX,YQ,NL,S,XQMX)
C
        IF(STRDL) GO TO 3ØØ
C
        IF(XQMX.GE.XMX) GO TO 2ØØ
        DYQ=2.Ø*DYQ
        YQØ=YQ(1)
        GO TO 4ØØ
C
2ØØ     STRDL=.TRUE.
C
3ØØ     IF(XQMX.LT.XMX) YQØ=YQ(1)
        IF(DYQ.LT.EPS) GO TO 5ØØ
        DYQ=Ø.5*DYQ
C
4ØØ   CONTINUE
C
C FILL IN ALL THE LEVELS FOR SYMMETRICAL DISTRIBUTIONS
5ØØ   IF(ISYM.EQ.Ø) GO TO 8ØØ
      K=NLEV-NL
      DO 6ØØ I=1,NL
        K=K+1
        YQ(K)=YQ(I)
6ØØ   CONTINUE
C
      K=NLEV
      DO 7ØØ I=1,NL
        YQ(I)=2.Ø*AM-YQ(K)
        K=K-1
7ØØ   CONTINUE
      IF(SYODD) YQ(NL+1)=AM
C
8ØØ   RETURN
      END
```

```
      SUBROUTINE QNTLEV(FØ,F1,XST,XMX,YQ,NLEV,S,XQMX)
C                                                   29/Ø5/78 P. KABAL
C THIS SUBROUTINE FINDS THE QUANTIZER OUTPUT LEVELS WHICH SATISFY
C THE NECESSARY CONDITIONS FOR A MINIMUM MEAN SQUARE ERROR QUANTIZER.
C IT USES FUNCTIONS FØ AND F1.
C
C FØ     - EXTERNAL FUNCTION TO CALCULATE THE CUMULATIVE PROBABILITY
C          DENSITY FUCNTION, I.E. THE INTEGRAL OF P(X) FROM MINUS
C          INFINITY TO Y, WHERE P(X) IS THE PROBABILITY DENSITY
C          FUNCTION.
C F1     - EXTERNAL FUNCTION TO CALCULATE THE INTEGRAL OF X*P(X)
C          FROM MINUS INFINITY TO Y, WHERE P(X) IS THE PROBABILITY
C          DENSITY FUNCTION
C XST    - INPUT VALUE SPECIFYING A STARTING VALUE FOR THE PROBABILITY
C          DENSITY FUNCTION. THIS VALUE SPECIFIES THE LOWER LIMIT
C          OF THE INTEGRALS USED BY THIS ROUTINE. NORMALLY XST IS A
C          NUMBER SUCH THAT FØ(XST)=F1(XST)=Ø.
C XMX    - X VALUE BEYOND WHICH THE PROBABILITY DENSITY FUNCTION
C          MAY BE CONSIDERED TO BE ZERO
C YQ     - INPUT ARRAY OF NLEV QUANTIZER OUTPUT VALUES. THE FIRST
C          ELEMENT OF YQ MUST BE SPECIFIED. THIS SUBROUTINE USES
C          YQ(1) AS A STARTING POINT AND CALCULATES THE REMAINING
C          ELEMENTS OF YQ SUCH THAT THEY SATISFY THE NECESSARY
C          CONDITIONS FOR A MINIMUM MEAN SQUARE ERROR QUANTIZER.
C          THE QUANTIZER BREAK POINTS ARE ASSUMED TO LIE MIDWAY
C          BETWEEN THE QUANTIZER OUTPUT LEVELS. THE QUANTIZER
C          OUTPUT LEVELS WILL BE IN ASCENDING ORDER.
C NLEV   - NUMBER OF QUANTIZER LEVELS
C S      - STANDARD DEVIATION OF THE PROBABILITY DENSITY FUNCTION.
C          THIS VALUE IS ONLY USED TO CALCULATE THE STARTING STEP
C          SIZE AND CONVERGENCE CRITERIA FOR THE ITERATIVE PROCEDURE
C          USED IN THIS ROUTINE (Ø.1*S AND 1E-7*S RESPECTIVELY).
C XQMX   - OUTPUT VALUE REPRESENTING THE LARGEST QUANTIZER BREAK POINT.
C          IF XQMX IS GREATER THAN XMX, YQ(1) IS TOO LARGE. IF XQMX IS
C          LESS THAN XMX, YQ(1) IS TOO SMALL.
C
      DIMENSION YQ(NLEV)
      LOGICAL STRDL
      DATA NIMX/1ØØ/
C
      YQQ=YQ(1)
      XSTT=XST
      EPS=1E-7*S
C
      DO 5ØØ I=1,NLEV
C
        STRDL=.FALSE.
        DXF=Ø.1*S
        XFØ=YQQ
        FØST=FØ(XSTT)
        F1ST=F1(XSTT)
C
          DO 4ØØ ITER=1,NIMX
C
            IF(XFØ.GT.XMX) GO TO 6ØØ
            XFIN=XFØ+DXF
            R=(F1(XFIN)-F1ST)-YQQ*(FØ(XFIN)-FØST)
C
            IF(STRDL) GO TO 3ØØ
            IF(R.GT.Ø.Ø) GO TO 2ØØ
            DXF=2.Ø*DXF
            XFØ=XFIN
            GO TO 4ØØ
C
2ØØ         STRDL=.TRUE.
C
3ØØ         IF(R.LE.Ø.Ø) XFØ=XFIN
            IF(DXF.LT.EPS) GO TO 45Ø
            DXF=Ø.5*DXF
C
4ØØ       CONTINUE
          GO TO 8ØØ
C
45Ø       IF(I.GE.NLEV) GO TO 8ØØ
          YQQ=2.Ø*XFIN-YQQ
          YQ(I+1)=YQQ
          XSTT=XFØ
C
5ØØ     CONTINUE
        GO TO 8ØØ
C
6ØØ     IF(I.GE.NLEV) GO TO 8ØØ
        I=I+1
        DO 7ØØ K=I,NLEV
          YQ(K)=XFØ
7ØØ     CONTINUE
C
8ØØ     XQMX=XFØ
        RETURN
        END
```

```
      SUBROUTINE QNTMSE(FØ,F1,F2,XMX,YQ,NLEV,RMSE)
C                                                18/Ø5/78 P. KABAL
C THIS SUBROUTINE CALCULATES THE RESULTING MEAN SQUARE ERROR FOR
C A QUANTIZER.  IT USES FUNCTIONS FØ, F1 AND F2.
C
C FØ     - EXTERNAL FUNCTION TO CALCULATE THE CUMULATIVE PROBABILITY
C          DENSITY FUNCTION, I.E. THE INTEGRAL OF P(X) FROM MINUS
C          INFINITY TO Y, WHERE P(X) IS THE PROBABILITY DENSITY
C          FUNCTION
C F1     - EXTERNAL FUNCTION TO CALCULATE THE INTEGRAL OF X*P(X)
C          FROM MINUS INFINITY TO Y, WHERE P(X) IS THE PROBABILITY
C          DENSITY FUNCTION
C F2     - EXTERNAL FUNCTION TO CALCULATE THE INTEGRAL OF X*X*P(X)
C          FROM MINUS INFINITY TO Y, WHERE P(X) IS THE PROBABILITY
C          DENSITY FUNCTION
C XMX    - X VALUE BEYOND WHICH THE PROBABILITY DENSITY FUNCTION
C          MAY BE CONSIDERED TO BE ZERO
C YQ     - INPUT ARRAY OF NLEV QUANTIZER OUTPUT LEVELS. THESE SHOULD
C          BE IN ASCENDING ORDER. THE QUANTIZER BREAK POINTS ARE
C          ASSUMED TO LIE MIDWAY BETWEEN THE QUANTIZER OUTPUT LEVELS.
C NLEV   - NUMBER OF QUANTIZER LEVELS
C RMSE   - RESULTING MEAN SQUARE ERROR FOR THE QUANTIZER
C
      DIMENSION YQ(NLEV)
C
      RMSE=Ø.Ø
      I=1
      YQI=YQ(1)
      FØST=Ø.Ø
      F1ST=Ø.Ø
      F2ST=Ø.Ø
C
1ØØ   XQ=XMX
      YQQ=YQI
      I=I+1
      IF(I.GT.NLEV) GO TO 2ØØ
      YQI=YQ(I)
      XQ=Ø.5*(YQQ+YQI)
C
2ØØ   FØFN=FØ(XQ)
      F1FN=F1(XQ)
      F2FN=F2(XQ)
C
      RMSE=RMSE+((F2FN-F2ST)-2.Ø*YQQ*(F1FN-F1ST)+YQQ*YQQ*(FØFN-FØST))
C
      FØST=FØFN
      F1ST=F1FN
      F2ST=F2FN
      IF(I.LE.NLEV) GO TO 1ØØ
C
      RETURN
C
      END
```

```
      SUBROUTINE QNTENT(FØ,XMX,YQ,NLEV,ENTRPY)
C                                                2Ø/Ø6/78 P. KABAL
C THIS SUBROUTINE CALCULATES THE RESULTING ENTROPY FOR A QUANTIZER.
C IT USES FUNCTION FØ.
C
C FØ     - EXTERNAL FUNCTION TO CALCULATE THE CUMULATIVE PROBABILITY
C          DENSITY FUNCTION, I.E. THE INTEGRAL OF P(X) FROM MINUS
C          INFINITY TO Y, WHERE P(X) IS THE PROBABILITY DENSITY
C          FUNCTION
C XMX    - X VALUE BEYOND WHICH THE PROBABILITY DENSITY FUNCTION
C          MAY BE CONSIDERED TO BE ZERO
C YQ     - INPUT ARRAY OF NLEV QUANTIZER OUTPUT LEVELS. THESE SHOULD
C          BE IN ASCENDING ORDER. THE QUANTIZER BREAK POINTS ARE
C          ASSUMED TO LIE MIDWAY BETWEEN THE QUANTIZER OUTPUT LEVELS.
C NLEV   - NUMBER OF QUANTIZER LEVELS
C ENTRPY - RESULTING ENTROPY FOR THE QUANTIZER
C
      DIMENSION YQ(NLEV)
      DATA C1/1.442695Ø41/
C
      ENTRPY=Ø.Ø
      I=1
      YQI=YQ(1)
      FØST=Ø.Ø
C
1ØØ   XQ=XMX
      I=I+1
      IF(I.GT.NLEV) GO TO 2ØØ
      YQQ=YQI
      YQI=YQ(I)
      XQ=Ø.5*(YQQ+YQI)
C
2ØØ   FØFN=FØ(XQ)
      PR=FØFN-FØST
      ENTRPY=ENTRPY-PR*ALOG(PR)
C
      FØST=FØFN
      IF(I.LE.NLEV) GO TO 1ØØ
C
      ENTRPY=C1*ENTRPY
      RETURN
C
      END
```

```
      FUNCTION GAMCDF(X,P)
C                                                   16/Ø5/78 P. KABAL
C THIS FUNCTION CALCULATES THE INTEGRAL FROM ZERO TO X OF THE ONE-SIDED
C GAMMA DENSITY FUNCTION WITH POSITIVE PARAMETER P. IT USES EITHER A
C CONTINUED FRACTION OR SERIES EXPANSION DEPENDING ON THE VALUES OF
C THE ARGUMENTS (SEE THE HANDBOOK OF MATHEMATICAL FUNCTIONS BY
C M. ABRAMOWITZ AND I.A. STEGUN, EQUATIONS 26.4.6 AND 26.4.1Ø). THIS
C ROUTINE CALLS FUNCTION ALGAMA TO CALCULATE THE LOG-GAMMA FUNCTION.
C
      DATA ALRG/1E18/,EPS/1E-8/,EXMN/-88.Ø/
C
      GAMCDF=Ø.Ø
      IF(X.LE.Ø.Ø) GO TO 9ØØ
C
      AT=P*ALOG(X)-(X+ALGAMA(P))
      AX=Ø.Ø
      IF(AT.GT.EXMN) AX=EXP(AT)
C
      IF(X.LE.1.Ø .OR. X.LT.P) GO TO 4ØØ
C
C CONTINUED FRACTION EXPANSION
C INITIALIZATION
      O=1.Ø-P
      BN=X+2.Ø-P
      AN=Ø.Ø
      AN1=1.Ø
      AD1=X
      AN2=X+1.Ø
      AD2=X*BN
      SP=AN2/AD2
C
C CALCULATE AN EXTENDED EXPANSION BY RECURSION
1ØØ   AN=AN+1.Ø
      BN=BN+2.Ø
      CN=AN*(AN+O)
      ANUM=AN2*BN-AN1*CN
      ADEN=AD2*BN-AD1*CN
C
C CHECK FOR CONVERGENCE
      IF(ADEN.EQ.Ø.Ø) GO TO 2ØØ
      S=ANUM/ADEN
      DF=ABS(SP-S)
      IF(DF.LE.EPS .AND. DF.LE.S*EPS) GO TO 3ØØ
      SP=S
C
2ØØ   AN1=AN2
      AN2=ANUM
      AD1=AD2
      AD2=ADEN
C
C SCALE THE TERMS TO PREVENT OVERFLOWS
      IF(ABS(ANUM).LT.ALRG) GO TO 1ØØ
      AN1=AN1/ALRG
      AN2=AN2/ALRG
      AD1=AD1/ALRG
      AD2=AD2/ALRG
C
3ØØ   GAMCDF=1.Ø-AX*S
      GO TO 9ØØ
C
C SERIES APPROXIMATION
4ØØ   CN=1.Ø
      S=1.Ø
      BN=P
C
5ØØ   BN=BN+1.Ø
      CN=CN*X/BN
      S=S+CN
      IF(CN.GT.EPS) GO TO 5ØØ
C
      GAMCDF=AX*S/P
C
9ØØ   RETURN
      END
      FUNCTION ALGAMA(X)
C                                                   16/Ø5/78 P. KABAL
C THIS FUNCION CALCULATES THE NATURAL LOGARITHM OF THE GAMMA FUNCTION.
C FOR ARGUMENT VALUES LESS THAN 1E-9, ALGAMA IS SET TO -1E38. THIS
C FUNCTION CALCULATES THE LOG-GAMMA FUNCTION USING EQUATION 6.1.41
C IN THE HANDBOOK OF MATHEMATICAL FUNCTIONS BY M. ABRAMOWITZ AND
C I.A. STEGUN.
C
      DOUBLE PRECISION Z,TERM,RZ,RZ2,C1,C2,C3,C4,C5,DLOG
      DATA C1/Ø.918938533Ø467274178DØ/,C2/8.3333333333333333333D-2/,
     *     C3/2.7777777777777777777D-3/,C4/7.936Ø79365Ø79365Ø79D-4/,
     *     C5/5.952380952380952381ØD-4/
C
      Z=X
      IF(X.GT.1E1Ø) GO TO 3Ø
2Ø    IF(X.GT.1E-9) GO TO 4Ø
      ALGAMA=-1E38
      GO TO 1ØØ
C
C FOR LARGE ARGUMENTS USE AN ABREVIATED SERIES
3Ø    ALGAMA=Z*(DLOG(Z)-1DØ)
      GO TO 1ØØ
C
C USE THE GAMMA FUNCTION RECURSION TO TELESCOPE THE ARGUMENT
C SO THAT IT IS LARGER THAN 18
4Ø    TERM=1DØ
5Ø    IF(Z.GT.18DØ) GO TO 6Ø
      TERM=TERM*Z
      Z=Z+1DØ
      GO TO 5Ø
C
C EULER-MCLAURIN SERIES EXPANSION (Z LARGER THAN 18)
6Ø    RZ=1DØ/Z
      RZ2=RZ*RZ
      ALGAMA=(Z-Ø.5DØ)*DLOG(Z)-Z+C1-DLOG(TERM)+
     *       RZ*(C2-(RZ2*(C3-(RZ2*(C4-RZ2*C5)))))
C
1ØØ   RETURN
```

```
      FUNCTION FGMØ(X)
C                                                 17/Ø5/78 P. KABAL
C THIS FUNCTION CALCULATES THE FOLLOWING FUNCTIONS.
C
C FOR THE TWO SIDED GAMMA FUNCTION P(U) WITH PARAMETER A AND STANDARD
C DEVIATION S,
C FGMØ  - INTEGRAL FROM MINUS INFINITY TO X OF P(U)
C FGM1  - INTEGRAL FROM MINUS INFINITY TO X OF U*P(U)
C FGM2  - INTEGRAL FROM MINUS INFINITY TO X OF U*U*P(U)
C
C FOR THE GAUSSIAN DENSITY FUNCTION P(U) WITH STANDARD DEVIATION S,
C FGSØ  - INTEGRAL FROM MINUS INFINITY TO X OF P(U)
C FGS1  - INTEGRAL FROM MINUS INFINITY TO X OF U*P(U)
C FGS2  - INTEGRAL FROM MINUS INFINITY TO X OF U*U*P(U)
C
C COMMON BLOCK FCN,
C    S   - STANDARD DEVIATION OF THE DENSITY FUNCTION
C    A   - PARAMETER FOR THE GAMMA DENSITY
C             A=Ø.5, DOUBLE SIDED GAMMA DENSITY
C             A=1.Ø, LAPLACIAN DENSITY
C    AL  - PARAMETER FOR THE GAMMA DENSITY
C             AL=SQRT(A*(A+1))/S
C
      COMMON /FCN/ S,A,AL
      DATA S,A,AL/1.Ø,1.Ø,1.414213562/,C1/Ø.3989422804/
C
C FGMØ
      AX=X*AL
      IF(AX.LT.Ø.Ø) AX=-AX
      FN=GAMCDF(AX,A)
      IF(X.LT.Ø.Ø) FN=-FN
      FGMØ=Ø.5*(1DØ+FN)
      RETURN
C
C FGM1
      ENTRY FGM1(X)
      AX=X*AL
      IF(AX.LT.Ø.Ø) AX=-AX
      FN=GAMCDF(AX,A+1.Ø)
      FGM1=(Ø.5*A/AL)*(FN-1DØ)
      RETURN
C
C FGM2
      ENTRY FGM2(X)
      AX=X*AL
      IF(AX.LT.Ø.Ø) AX=-AX
      FN=GAMCDF(AX,A+2.Ø)
      IF(X.LT.Ø.Ø) FN=-FN
      FGM2=(Ø.5*S*S)*(1DØ+FN)
      RETURN
C
C FGSØ
      ENTRY FGSØ(X)
      AX=X/S
      FN=GAMCDF(Ø.5*AX*AX,Ø.5)
      IF(X.LT.Ø.Ø) FN=-FN
      FGSØ=Ø.5*(1DØ+FN)
      RETURN
C
C FGS1
      ENTRY FGS1(X)
      AX=X/S
      FN=GAMCDF(Ø.5*AX*AX,1.Ø)
      FGS1=C1*(FN-1DØ)
      RETURN
C
C FGS2
      ENTRY FGS2(X)
      AX=X/S
      FN=GAMCDF(Ø.5*AX*AX,1.5)
      IF(X.LT.Ø.Ø) FN=-FN
      FGS2=(Ø.5*S*S)*(1DØ+FN)
      RETURN
C
      END
```

Appendix D

Program Listing

This Appendix includes a listing for a routine to calculate the minimum mean square error for a signal whose probability density function is specified in tabulated form. Intermediate values for the probability density function are determined by linear interpolation.

```
      SUBROUTINE QNTDSG(P,X,N,YQ,NLEV,IQTYPE,PARM,RMSE,ENTRPY)
C                                                21/06/78 P. KABAL
C  THIS SUBROUTINE DESIGNS QUANTIZERS. IT USES SUBROUTINES OPTQNT,
C  PDFCN, QNTLEV, CUBIC, QNTMSE, AND QNTENT.
C
C  P      - WORK ARRAY. THIS ARRAY IS USED TO STORE SAMPLES OF THE
C           PROBABILITY DENSITY FUNCTION. SEE THE DESCRIPTION OF THE
C           PARAMETER N BELOW.
C  X      - WORK ARRAY. THIS ARRAY IS USED TO STORE THE ABSCISSA VALUES
C           CORRESPONDING TO THE VALUES OF THE PROBABILITY DENSITY
C           FUNCTION STORED IN ARRAY P. SEE THE DESCRIPTION OF THE
C           PARAMETER N BELOW.
C  N      - NUMBER OF ELEMENTS IN P AND X. ARRAYS P AND X ARE WORK
C           ARRAYS USED ONLY FOR THE MINIMUM MEAN SQUARE ERROR DESIGNS
C           (IQTYPE EQUAL TO 'N', 'L' OR 'G'). A SUGGESTED VALUE FOR
C           N IN THESE CASES IS 200 TO 500.
C  YQ     - OUTPUT ARRAY OF NLEV QUANTIZER OUTPUT VALUES. THE QUANTIZER
C           BREAK POINTS OCCUR MIDWAY BETWEEN THE QUANTIZER OUTPUT
C           LEVELS. THE QUANTIZER OUTPUT LEVELS ARE IN ASCENDING ORDER.
C  NLEV   - NUMBER OF QUANTIZER LEVELS
C  IQTYPE - CHARACTER DESIGNATING THE TYPE OF QUANTIZER DESIRED.
C           'A' - A-LAW QUANTIZER
C           'M' - MU-LAW QUANTIZER
C           'U' - UNIFORM QUANTIZER
C           'N' - MINIMUM MEAN SQUARE ERROR QUANTIZER FOR A ZERO MEAN
C                 GAUSSIAN (NORMAL) PROBABILITY DENSITY FUNCTION
C           'L' - MINIMUM MEAN SQUARE ERROR QUANTIZER FOR A ZERO MEAN
C                 LAPLACIAN PROBABILITY DENSITY FUNCTION
C           'G' - MINIMUM MEAN SQUARE ERROR QUANTIZER FOR A ZERO MEAN
C                 GAMMA PROBABILITY DENSITY FUNCTION
C  PARM   - INPUT PARAMETER, THIS PARAMETER HAS DIFFERENT MEANINGS
C           FOR THE DIFFERENT VALUES OF IQTYPE.
C           'A' - PARAMETER A OF THE A-LAW QUANTIZER. THE QUANTIZER
C                 INPUT IS ASSUMED TO LIE BETWEEN -1 AND +1.
C           'M' - PARAMETER MU OF THE MU-LAW QUANTIZER. THE QUANTIZER
C                 INPUT IS ASSUMED TO LIE BETWEEN -1 AND +1.
C           'U' - THE MAXIMUM VALUE OF THE QUANTIZER INPUT (ASSUMED
C                 SYMMETRIC)
C           'N' - STANDARD DEVIATION OF THE GAUSSIAN DENSITY FUNCTION
C           'L' - STANDARD DEVIATION OF THE LAPLACIAN DENSITY FUNCTION
C           'G' - STANDARD DEVIATION OF THE GAMMA DENSITY FUNCTION
C  RMSE   - RESULTING MEAN SQUARE ERROR. THIS PARAMETER IS ONLY USED
C           FOR THE MINIMUM MEAN SQUARE ERROR QUANTIZERS (IQTYPE
C           EQUAL TO 'U', 'N', 'L', OR 'G').
C  ENTPRY - RESULTING ENTROPY. THIS PARAMETER IS ONLY USED FOR THE
C           MINIMUM MEAN SQUARE ERROR QUANTIZERS (IQTYPE EQUAL TO 'U',
C           'N', 'L', OR 'G').
C
      BYTE IQTYPE,ALAW,MULAW,UNIF,GAUSS,LAPLC,GAMMA
      DIMENSION P(N),X(N),YQ(NLEV)
      DATA ALAW,MULAW,UNIF/'A','M','U'/,GAUSS,LAPLC,GAMMA/'N','L','G'/
      DATA SQPI/0.3989422804/,RT2I/0.7071067812/,RT2M/-1.414213562/,
     *     RT38/0.06891611194/,RT3M/-0.8660254038/,ISYM/1/,
     *     C1/1.442695041/
C
      IF(IQTYPE.EQ.GAUSS .OR. IQTYPE.EQ.LAPLC .OR. IQTYPE.EQ.GAMMA)
     *    GO TO 500
C
C  A-LAW, MU-LAW OR UNIFORM QUANTIZER
      RMSE=0.0
      ENTRPY=0.0
      XMAX=1.0
      IF(IQTYPE.EQ.UNIF) XMAX=PARM
      FN=NLEV
      DX=2.0*XMAX/FN
      XOFFS=XMAX+0.5*DX
C
C  GENERATE A UNIFORM QUANTIZER
      DO 100 I=1,NLEV
        YQ(I)=FLOAT(I)*DX-XOFFS
100   CONTINUE
      IF( IQTYPE.NE.UNIF) GO TO 150
      RMSE=DX*DX/12.0
      ENTRPY=C1*ALOG(FN)
C
C  MU-LAW QUANTIZER
150   IF(IQTYPE.NE.MULAW) GO TO 300
      CONS=ALOG(1.0+PARM)
      DO 200 I=1,NLEV
        YQI=YQ(I)
        YQQ=(EXP(CONS*ABS(YQI))-1.0)/PARM
        IF(YQI.LT.0.0) YQQ=-YQQ
        YQ(I)=YQQ
200   CONTINUE
      GO TO 800
C
C  A-LAW QUANTIZER
300   IF(IQTYPE.NE.ALAW) GO TO 800
      CONS=1.0+ALOG(PARM)
      DO 400 I=1,NLEV
        YQI=YQ(I)
        CYQI=CONS*ABS(YQI)
        IF(CYQI.LE.1.0) YQQ=CYQI
        IF(CYQI.GT.1.0) YQQ=EXP(CYQI-1.0)
        IF(YQI.LT.0.0) YQQ=-YQQ
        YQ(I)=YQQ/PARM
400   CONTINUE
      GO TO 800
```

```
C
C  MINIMUM MEAN SQUARE ERROR QUANTIZERS
5ØØ     S=PARM
        IF(IQTYPE.NE.GAMMA) GO TO 51Ø
        SMX=15.Ø
        DXMX=1ØØ.Ø/FLOAT(N)
        IF(DXMX.GT.Ø.5) DXMX=Ø.5
        X(1)=Ø.Ø
        XI=1E-3*S
        IST=2
        GO TO 53Ø
C
51Ø     IST=1
        XI=Ø.Ø
        IF(IQTYPE.NE.GAUSS) GO TO 52Ø
        SMX=5.Ø
        DXMX=2.5/FLOAT(N)
        GO TO 53Ø
C
52Ø     SMX=1Ø.Ø
        DXMX=5.Ø/FLOAT(N)
C
53Ø     R=(SMX-DXMX/(SMX/(SMX-DXMX))**(N-2))/(SMX-DXMX)
        DX=S*SMX*(R-1.Ø)/(R**(N-1)-1.Ø)
C
C  FILL IN THE ARRAYS P AND X
        DO 55Ø I=IST,N
          XIS=XI/S
          X(I)=XI
          IF(IQTYPE.EQ.GAUSS) PI=SQPI*EXP(-Ø.5*XIS*XIS)
          IF(IQTYPE.EQ.GAMMA) PI=SQRT(RT38/XIS)*EXP(RT3M*XIS)
          IF(IQTYPE.EQ.LAPLC) PI=RT2I*EXP(RT2M*XIS)
          P(I)=PI/S
          XI=XI+DX
          DX=R*DX
55Ø     CONTINUE
        IF(IQTYPE.EQ.GAMMA) P(1)=P(2)
C
6ØØ     CALL OPTQNT(P,X,N,ISYM,YQ,NLEV)
        CALL QNTMSE(P,X,N,YQ,NLEV,RMSE)
        RMSE=2.Ø*RMSE
        CALL QNTENT(P,X,N,YQ,NLEV,ENTRPY)
        ENTRPY=2.Ø*ENTRPY
C
8ØØ     RETURN
        END
```

```
      SUBROUTINE OPTQNT(P,X,N,ISYM,YQ,NLEV)
C                                                   Ø9/Ø6/78 P. KABAL
C THIS SUBROUTINE ITERATES TO FIND THE QUANTIZER WHICH MINIMIZES THE
C MEAN SQUARE ERROR FOR A GIVEN PROBABILITY DENSITY FUNCTION. IT USES
C SUBROUTINES PDFCN, QNTLEV AND CUBIC.
C
C P       - ARRAY OF SAMPLES OF THE PROBABILITY DENSITY FUNCTION.
C           THE PROBABILITY DENSITY FUNCTION IS LINEARLY INTERPOLATED
C           BETWEEN THE GIVEN VALUES.
C X       - ARRAY OF ABSCISSA VALUES CORRESPONDING TO THE TABULATED
C           VALUES OF THE PROBABILITY DENSITY FUNCTION IN THE ARRAY P.
C           THE ABSCISSA VALUES SHOULD BE MONOTONICALLY INCREASING.
C N       - NUMBER OF ELEMENTS IN P AND X (AT LEAST 2)
C ISYM    - INPUT CONTROL VARIABLE
C           IF ZERO, THE FULL RANGE OF THE PROBABILITY DENSITY FUNCTION
C                   MUST BE SPECIFIED
C           IF NONZERO, THE PROBABILITY DENSITY FUNCTION IS ASSUMED TO
C                   SYMMETRIC ABOUT ITS MEAN. IN THIS CASE ONLY THE
C                   UPPER HALF OF THE PROBABILITY DENSITY FUNCTION IS
C                   SPECIFIED. X(1) IS ASSUMED TO BE THE MEAN OF THE
C                   PROBABILITY DENSITY FUNCTION.
C YQ      - OUTPUT ARRAY OF NLEV QUANTIZER OUTPUT VALUES. THE QUANTIZER
C           BREAK POINTS OCCUR MIDWAY BETWEEN THE QUANTIZER OUTPUT
C           LEVELS. THE QUANTIZER OUTPUT LEVELS ARE IN ASCENDING ORDER.
C NLEV    - NUMBER OF QUANTIZER LEVELS
C
      DIMENSION P(N),X(N),YQ(NLEV)
      LOGICAL STRDL,SYODD
      DATA NIMX/1ØØ/
C
C CALCULATE THE MEAN AND VARIANCE OF THE DISTRIBUTION
      CALL PDFCN(P,X,N,ISYM,PX,PMEAN,PVAR)
      S=SQRT(PVAR)
C
C INITIALIZATION, CHECK FOR A SYMMETRICAL DISTRIBUTION
      YQ(1)=PMEAN
      IF(NLEV.LE.1) GO TO 8ØØ
      YQØ=X(1)
      STRDL=.FALSE.
      DYQ=Ø.5*S
      EPS=1E-5*S
      XST=YQØ
      NL=NLEV
      IF(ISYM.NE.Ø) NL=NL/2
      SYODD=ISYM.NE.Ø .AND. 2*NL.NE.NLEV
```

```
C
C   ITERATE TO FIND A MINIMUM MEAN SQUARE ERROR QUANTIZER
        DO 400 ITER=1,NIMX
          YQ(1)=YQ0+DYQ
          IF(SYODD) XST=0.5*(PMEAN+YQ(1))
          CALL QNTLEV(P,X,N,XST,YQ,NL,XQMX)
C
          IF(STRDL) GO TO 300
C
          IF(XQMX.GE.X(N)) GO TO 200
          DYQ=2.0*DYQ
          YQ0=YQ(1)
          GO TO 400
C
200       STRDL=.TRUE.
C
300       IF(XQMX.LT.X(N)) YQ0=YQ(1)
          IF(DYQ.LT.EPS) GO TO 500
          DYQ=0.5*DYQ
C
400     CONTINUE
C
C   FILL IN ALL THE LEVELS FOR SYMMETRICAL DISTRIBUTIONS
500     IF(ISYM.EQ.0) GO TO 800
        K=NLEV-NL
        DO 600 I=1,NL
          K=K+1
          YQ(K)=YQ(I)
600     CONTINUE
C
        K=NLEV
        DO 700 I=1,NL
          YQ(I)=2.0*PMEAN-YQ(K)
          K=K-1
700     CONTINUE
        IF(SYODD) YQ(NL+1)=PMEAN
C
800     RETURN
        END
```

```
      SUBROUTINE QNTLEV(P,X,N,XST,YQ,NLEV,XQMX)
C                                                      20/06/78 P. KABAL
C THIS SUBROUTINE FINDS THE QUANTIZER OUTPUT LEVELS WHICH SATISFY
C THE NECESSARY CONDITIONS FOR A MINIMUM MEAN SQUARE ERROR QUANTIZER.
C IT USES SUBROUTINE CUBIC.
C
C P       - ARRAY OF SAMPLES OF THE PROBABILITY DENSITY FUNCTION.
C           THE PROBABILITY DENSITY FUNCTION IS LINEARLY INTERPOLATED
C           BETWEEN THE GIVEN VALUES.
C X       - ARRAY OF ABSCISSA VALUES CORRESPONDING TO THE TABULATED
C           VALUES OF THE PROBABILITY DENSITY FUNCTION IN THE ARRAY P.
C           THE ABSCISSA VALUES SHOULD BE MONOTONICALLY INCREASING.
C N       - NUMBER OF ELEMENTS IN P AND X (AT LEAST 2)
C XST     - INPUT VALUE SPECIFYING A STARTING VALUE FOR THE
C           PROBABILITY DENSITY FUNCTION. THE PROBABILITY DENSITY
C           FUNCTION IS ASSUMED TO BE ZERO FOR ABSCISSA VALUES LESS
C           THAN XST. NORMALLY XST=X(1).
C YQ      - INPUT ARRAY OF NLEV QUANTIZER OUTPUT VALUES. THE FIRST
C           ELEMENT OF YQ MUST BE SPECIFIED. THIS SUBROUTINE USES
C           YQ(1) AS A STARTING POINT AND CALCULATES THE REMAINING
C           ELEMENTS OF YQ SUCH THAT THEY SATISFY THE NECESSARY
C           CONDITIONS FOR A MINIMUM MEAN SQUARE ERROR QUANTIZER.
C           THE QUANTIZER BREAK POINTS ARE ASSUMED TO LIE MIDWAY
C           BETWEEN THE QUANTIZER OUTPUT LEVELS. THE QUANTIZER
C           OUTPUT LEVELS WILL BE IN ASCENDING ORDER.
C NLEV    - NUMBER OF QUANTIZER LEVELS
C XQMX    - LARGEST QUANTIZER BREAK POINT. IF XQMX IS EQUAL TO X(N),
C           YQ(1) IS TOO LARGE. IF XQMX IS LESS THAN X(N), YQ(1)
C           IS TOO SMALL.
C
      DIMENSION P(N),X(N),YQ(NLEV),ZR(3),ZI(3),C(4)
      EQUIVALENCE (C(3),AYQ),(C(4),A2)
      DATA C(2)/0.0/
C
      S=0.0
      YQQ=YQ(1)
      XQMX=YQQ
      J=2
      PIM1=P(1)
      XIM1=X(1)
      XSTT=XST
```

```
C
      DO 5ØØ I=2,N
        PI=P(I)
        XI=X(I)
        IF(XI.LE.XSTT) GO TO 45Ø
        XFIN=XI
        A=(PI-PIM1)/(XI-XIM1)
        A2=A+A
        B=PIM1-A*XIM1
C
1ØØ     AYQ=3.Ø*(A*YQQ+B)
        EF=XFIN-YQQ
        ES=XSTT-YQQ
        T1=EF*EF*(A2*EF+AYQ)
        T2=ES*ES*(A2*ES+AYQ)
        S1=S+(T1-T2)
C
        IF(S1.LE.Ø.Ø) GO TO 4ØØ
C
        C(1)=S-T2
        CALL CUBIC(C,ZR,ZI,NR)
        DO 2ØØ K=1,NR
          IF(ZI(K).NE.Ø.Ø) GO TO 2ØØ
          EQ=ZR(K)
          IF(EQ.LT.Ø.Ø) GO TO 2ØØ
          XQ=EQ+YQQ
          IF(XQ.GE.XSTT .AND. XQ.LT.XFIN) GO TO 3ØØ
2ØØ     CONTINUE
C
3ØØ     XQMX=XQ
        IF(J.GT.NLEV) GO TO 8ØØ
        YQQ=2.Ø*XQ-YQQ
        YQ(J)=YQQ
        J=J+1
        S=Ø.Ø
        XSTT=XQ
        GO TO 1ØØ
C
4ØØ     XSTT=XFIN
        S=S1
45Ø     PIM1=PI
        XIM1=XI
5ØØ   CONTINUE
C
C
      XQMX=X(N)
      IF(J.GT.NLEV) GO TO 8ØØ
      DO 7ØØ K=J,NLEV
        YQ(K)=X(N)
7ØØ   CONTINUE
C
8ØØ   RETURN
      END
```

```
      SUBROUTINE QNTMSE(P,X,N,YQ,NLEV,RMSE)
C                                                   2Ø/Ø6/78 P. KABAL
C THIS SUBROUTINE CALCULATES THE RESULTING MEAN SQUARE ERROR FOR
C A QUANTIZER. THE PROBABILITY DENSITY FUNCTION IS LINEARLY
C INTERPOLATED BETWEEN GIVEN VALUES.
C
C P      - ARRAY OF SAMPLES OF THE PROBABILITY DENSITY FUNCTION.
C X      - ARRAY OF ABSCISSA VALUES CORRESPONDING TO THE TABULATED
C          VALUES OF THE PROBABILITY DENSITY FUNCTION IN THE ARRAY P.
C          THE ABSCISSA VALUES SHOULD BE MONOTONICALLY INCREASING.
C N      - NUMBER OF ELEMENTS IN P AND X (AT LEAST 2)
C YQ     - INPUT ARRAY OF NLEV QUANTIZER OUTPUT LEVELS. THESE SHOULD
C          BE IN ASCENDING ORDER. THE QUANTIZER BREAK POINTS ARE
C          ASSUMED TO LIE MIDWAY BETWEEN THE QUANTIZER OUTPUT LEVELS.
C NLEV   - NUMBER OF QUANTIZER LEVELS
C RMSE   - RESULTING MEAN SQUARE ERROR FOR THE QUANTIZER
C
      DIMENSION P(N),X(N),YQ(NLEV)
C
      RMSE=Ø.Ø
      PIM1=P(1)
      XIM1=X(1)
      XST=XIM1
      XNP=1.Ø1*X(N)
      J=1
      YQQ=YQ(1)
      XQ=XNP
      IF(NLEV.LE.1) GO TO 1ØØ
      J=J+1
      YQJ=YQ(J)
      XQ=Ø.5*(YQQ+YQJ)
C
1ØØ   DO 5ØØ I=2,N
        PI=P(I)
        XI=X(I)
C
2ØØ     XFIN=XI
        IF(XI.GT.XQ) XFIN=XQ
        IF(XFIN.LE.XST) GO TO 3ØØ
C
        EF=XFIN-YQQ
        ES=XST-YQQ
        SE2=EF*EF+ES*ES
        RMSE=RMSE+((XFIN-XST)/(XI-XIM1))*(Ø.75*(PI-PIM1)*(EF+ES)*SE2+
     *             (SE2+EF*ES)*((XI-YQQ)*PIM1-(XIM1-YQQ)*PI))
C
        XST=XFIN
        IF(XI.LE.XQ) GO TO 4ØØ
```

```
C
300       XQ=XNP
          YQQ=YQJ
          IF(J.GE.NLEV) GO TO 200
          J=J+1
          YQJ=YQ(J)
          XQ=0.5*(YQQ+YQJ)
          GO TO 200
C
400       PIM1=PI
          XIM1=XI
500     CONTINUE
C
        RMSE=RMSE/3.0
        RETURN
        END


        SUBROUTINE QNTENT(P,X,N,YQ,NLEV,ENTRPY)
C                                                  20/06/78 P. KABAL
C  THIS SUBROUTINE CALCULATES THE RESULTING ENTROPY FOR A QUANTIZER.
C  THE PROBABILITY DENSITY FUNCTION IS LINEARLY INTERPOLATED BETWEEN
C  GIVEN VALUES.
C
C  P      - ARRAY OF SAMPLES OF THE PROBABILITY DENSITY FUNCTION.
C  X      - ARRAY OF ABSCISSA VALUES CORRESPONDING TO THE TABULATED
C           VALUES OF THE PROBABILITY DENSITY FUNCTION IN THE ARRAY P.
C           THE ABSCISSA VALUES SHOULD BE MONOTONICALLY INCREASING.
C  N      - NUMBER OF ELEMENTS IN P AND X (AT LEAST 2)
C  YQ     - INPUT ARRAY OF NLEV QUANTIZER OUTPUT LEVELS. THESE SHOULD
C           BE IN ASCENDING ORDER. THE QUANTIZER BREAK POINTS ARE
C           ASSUMED TO LIE MIDWAY BETWEEN THE QUANTIZER OUTPUT LEVELS.
C  NLEV   - NUMBER OF QUANTIZER LEVELS
C  ENTRPY - RESULTING ENTROPY FOR THE QUANTIZER
C
        DIMENSION P(N),X(N),YQ(NLEV)
        DATA C1/1.442695041/
C
        ENTRPY=0.0
        PRI=0.0
        PIM1=P(1)
        XIM1=X(1)
        XST=XIM1
        XNP=1.01*X(N)
        J=1
        YQQ=YQ(1)
        XQ=XNP
        IF(NLEV.LE.1) GO TO 100
        J=J+1
        YQJ=YQ(J)
        XQ=0.5*(YQQ+YQJ)
C
100     DO 500 I=2,N
          PI=P(I)
          XI=X(I)
C
200       XFIN=XI
          IF(XI.GT.XQ) XFIN=XQ
          IF(XFIN.LE.XST) GO TO 300
C
          PRI=PRI+((XFIN-XST)/(XI-XIM1))*((PI-PIM1)*(0.5*(XFIN+XST)-YQQ)+
     *             ((XI-YQQ)*PIM1-(XIM1-YQQ)*PI))
C
          XST=XFIN
          IF(XI.LE.XQ) GO TO 400
C
300       IF(PRI.GT.0.0) ENTRPY=ENTRPY-PRI*ALOG(PRI)
          PRI=0.0
          XQ=XNP
          YQQ=YQJ
          IF(J.GE.NLEV) GO TO 200
          J=J+1
          YQJ=YQ(J)
          XQ=0.5*(YQQ+YQJ)
          GO TO 200
C
400       PIM1=PI
          XIM1=XI
500     CONTINUE
C
        IF(PRI.GT.0.0) ENTRPY=ENTRPY-PRI*ALOG(PRI)
        ENTRPY=C1*ENTRPY
        RETURN
        END
```

```
      SUBROUTINE PDFCN(P,X,N,ISYM,PX,PMEAN,PVAR)
C                                                    23/04/78 P. KABAL
C THIS SUBROUTINE CALCULATES THE MEAN AND VARIANCE OF A TABULATED
C PROBABILITY DENSITY FUNCTION. THE PROBABILITY DENSITY FUNCTION IS
C LINEARLY INTERPOLATED BETWEEN THE GIVEN VALUES.
C
C P      - ARRAY OF SAMPLES OF THE PROBABILITY DENSITY FUNCTION.
C X      - ARRAY OF ABSCISSA VALUES CORRESPONDING TO THE TABULATED
C          VALUES OF THE PROBABILITY DENSITY FUNCTION IN THE ARRAY P.
C          THE ABSCISSA VALUES SHOULD BE MONOTONICALLY INCREASING.
C N      - NUMBER OF ELEMENTS IN P AND X (AT LEAST 2)
C ISYM   - INPUT CONTROL VARIABLE
C          IF ZERO, THE FULL RANGE OF THE PROBABILITY DENSITY FUNCTION
C                MUST BE SPECIFIED
C          IF NONZERO, THE PROBABILITY DENSITY FUNCTION IS ASSUMED TO
C                SYMMETRIC ABOUT ITS MEAN. IN THIS CASE ONLY THE
C                UPPER HALF OF THE PROBABILITY DENSITY FUNCTION IS
C                SPECIFIED. X(1) IS ASSUMED TO BE THE MEAN OF THE
C                PROBABILITY DENSITY FUNCTION.
C PX     - INTEGRAL OF P(X). THE VALUE OF PX REPRESENTS THE
C          NORMALIZATION FACTOR THAT SHOULD BE APPLIED TO THE VALUES
C          IN THE ARRAY P IN ORDER TO MAKE THE INTEGRAL UNITY.
C PMEAN  - MEAN OF THE DENSITY FUNCTION. PMEAN HAS ALREADY BEEN
C          NORMALIZED BY PX.
C PVAR   - VARIANCE OF THE DENSITY FUNCTION. PVAR HAS ALREADY BEEN
C          NORMALIZED BY PX.
C
      DIMENSION P(N),X(N)
C
      C1=0.0
      C2=0.0
      C3=0.0
      PIM1=P(1)
      XM=X(1)
      IF(ISYM.EQ.0) XM=0.5*(X(1)+X(N))
      XIM1=X(1)-XM
C
      DO 100 I=2,N
        PI=P(I)
        XI=X(I)-XM
C
        DP=PI-PIM1
        DXP=XI*PIM1-XIM1*PI
        SX=XI+XIM1
        SX2=XI*XI+XIM1*XIM1
        SXX=SX2+XI*XIM1
C
        C1=C1+(DP*SX+2.0*DXP)
        IF(ISYM.EQ.0) C2=C2+(2.0*DP*SXX+3.0*DXP*SX)
        C3=C3+(3.0*DP*SX*SX2+4.0*DXP*SXX)
C
        PIM1=PI
        XIM1=XI
100   CONTINUE
C
      IF(ISYM.EQ.0) GO TO 200
      PX=C1
      PMEAN=X(1)
      PVAR=C3/(6.0*PX)
      GO TO 300
C
200   PX=0.5*C1
      C2=C2/(6.0*PX)
      PMEAN=C2+XM
      PVAR=C3/(12.0*PX)-C2*C2
C
300   RETURN
      END
```

```
      SUBROUTINE CUBIC(A,ZR,ZI,N)
C                                                    13/04/78 P. KABAL
C THIS SUBROUTINE FINDS THE ROOTS OF A CUBIC EQUATION.
C
C A(4)*Z**3 + A(3)*Z**2 + A(2)*Z + A(1) = 0,
C
C WHERE THE COEFFICIENTS ARE REAL.
C
C A      - INPUT ARRAY OF FOUR COEFFICIENTS, A(1) IS THE CONSTANT
C          TERM. LEADING COEFFICIENTS OF THE POLYNOMIAL MAY BE ZERO.
C ZR     - OUTPUT ARRAY CONTAINING THE REAL PARTS OF THE N ROOTS,
C          WHERE N IS THREE OR LESS.
C ZI     - OUTPUT ARRAY CONTAINING THE IMAGINARY PARTS OF THE N ROOTS,
C          WHERE N IS THREE OR LESS.
C N      - NUMBER OF ROOTS FOUND. N MAY BE LESS THAN THREE IF THE
C          LEADING COEFFICIENTS OF THE POLYNOMIAL ARE ZERO.
C
      DIMENSION A(4),ZR(1),ZI(1)
      DATA RT3/0.8660254038/,TWP3/2.094395102/
C
      A4=A(4)
      IF(A4.EQ.0.0) GO TO 200
C
C TRANSFORM THE CUBIC TO X**3 + 3*Q*X - 2*R = 0,
C WHERE X = Z + A(3)/(3*A(4))
      A3=A(3)/A4
      A2=A(2)/A4
      A1=A(1)/A4
      A3B3=0.3333333333*A3
      Q=0.3333333333*A2-A3B3*A3B3
      R=0.1666666667*A2*A3-0.5*A1-A3B3*A3B3*A3B3
      N=3
      D=Q*Q*Q+R*R
      IF(D.LT.0.0) GO TO 100
```

```
C
C  ONE REAL ROOT AND A COMPLEX PAIR
        SQD=SQRT(D)
        T=R+SQD
        S1=ABS(T)**Ø.3333333333
        IF(T.LT.Ø.Ø) S1=-S1
        T=R-SQD
        S2=ABS(T)**Ø.3333333333
        IF(T.LT.Ø.Ø) S2=-S2
C
        ZR(1)=S1+S2-A3B3
        ZI(1)=Ø.Ø
        ZR(2)=-Ø.5*(S1+S2)-A3B3
        ZI(2)=RT3*(S1-S2)
        ZR(3)=ZR(2)
        ZI(3)=-ZI(2)
        GO TO 9ØØ
C
C  THREE REAL ROOTS
1ØØ     THT3=Ø.3333333333*ATAN2(SQRT(-D),R)
        TWQ=2.Ø*SQRT(-Q)
        ZR(1)=TWQ*COS(THT3)-A3B3
        ZI(1)=Ø.Ø
        ZR(2)=TWQ*COS(THT3+TWP3)-A3B3
        ZI(2)=Ø.Ø
        ZR(3)=TWQ*COS(THT3-TWP3)-A3B3
        ZI(3)=Ø.Ø
        GO TO 9ØØ
C
C  QUADRATIC EQUATION
2ØØ     A3=A(3)
        IF(A3.EQ.Ø.Ø) GO TO 4ØØ
        A2=A(2)/A3
        A1=A(1)/A3
        N=2
        D=A2*A2-4.Ø*A1
        IF(D.GT.Ø.Ø) GO TO 3ØØ
C
C  TWO COMPLEX ROOTS
        ZR(1)=-Ø.5*A2
        ZI(1)=Ø.5*SQRT(-D)
        ZR(2)=ZR(1)
        ZI(2)=-ZI(1)
        GO TO 9ØØ
C
C  TWO REAL ROOTS
3ØØ     SQD=SQRT(D)
        ZR(1)=-Ø.5*(A2-SQD)
        ZI(1)=Ø.Ø
        ZR(2)=-Ø.5*(A2+SQD)
        ZI(2)=Ø.Ø
        GO TO 9ØØ
C
C  LINEAR EQUATION, SINGLE REAL ROOT
4ØØ     IF(A(2).EQ.Ø.Ø) GO TO 5ØØ
        N=1
        ZR(1)=-A(1)/A(2)
        ZI(1)=Ø.Ø
        GO TO 9ØØ
C
C  NO ROOTS
5ØØ     N=Ø
C
C
9ØØ     RETURN
        END
```

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