Ultraspherical Windows

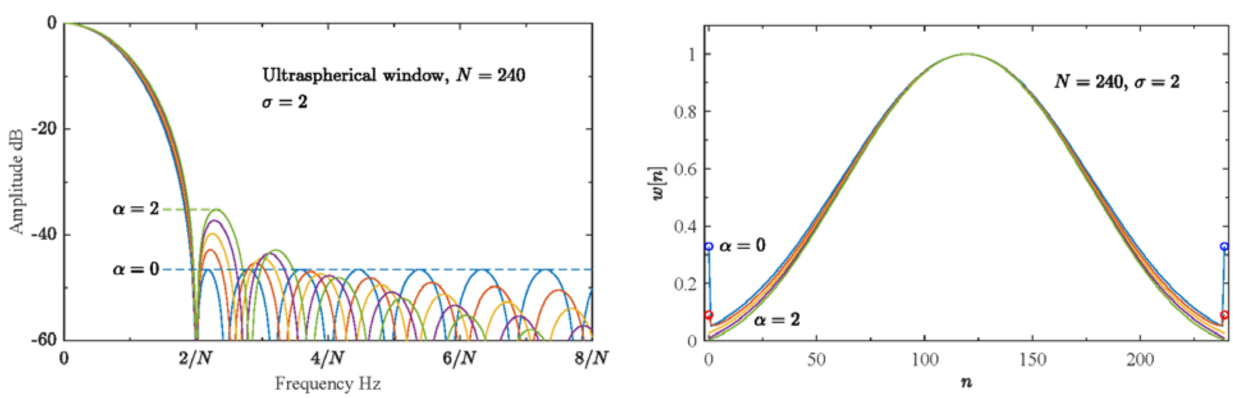
This package includes routines to calculate discrete-time and sampled continuous-time ultraspherical windows. The procedures are documented in a technical report included in the `Documents` folder. This report is also available at

> P. Kabal, "Ultraspherical Windows: Properties and Computation", Technical Report, Dept. Electrical & Computer Engineering, McGill University, April 2025 (available on-line at www.mmsp.ece.mcgill.ca).

Discrete-Time Ultraspherical Windows:

A DT ultraspherical window with N coefficients is characterized by two parameters α and x_0. The first controls the rate at which the spectral lobes decrease and the second determines the trade-off between the width of the main lobe of the spectrum and the sidelobe attenuation. For $\alpha = 0$, these are the Dolph-Chebyshev windows such as calculated by the Matlab routine `chebwin`. These windows have a frequency response which is equiripple in the stopband. For $\alpha = 1$, these are Saramäki windows which tend to minimize the fraction of energy in the stopband. The ultraspherical windows allow for non-integer values of α.

Some examples of ultraspherical windows are shown below. The left plot has the frequency responses and the right plot has the windows.



For a given N and α, there are two routines to find x_0:

- `USWinRdB` – Find x_0 for a sidelobe attenuation of R measured in dB.
- `USWinMLW` – Find x_0 for a main lobe width given by σ, where σ is the main lobe width relative to that of a rectangular window.

`USWin` -- Find the DT window coefficients for an ultraspherical window parameterized by N, x_0, and α. This routine calculates the window coefficients as the inverse DFT of the sampled spectrum. This is the "traditional" approach to calculating the windows.

Other routines:

- FindUSWinRdB – Find the sidelobe attenuation for an ultraspherical window with parameters N, x_0, and α.
- FindUSWinMLW – Find the main lobe width σ for an ultraspherical window with parameters N, x_0, and α.
- USPoly – Evaluate the ultraspherical polynomial given by N, and α (used by USWin).

extras Folder

The folder extras includes alternate routines for calculating the window coefficients.

- USWinR1 – Window calculated using the basic ultraspherical polynomial recurrence.
- USWinR1A – Window calculated using a modified recurrence.
- USWinR2 – Window calculated using a double-step recurrence.
- USChebyWinR2 – A simplified version of the double-step recurrence for a Dolph-Chebyshev window ($\alpha = 0$) and a Saramäki window ($\alpha = 1$).

The simplest of these recurrences is used in USWinR1 where the window is calculated with the single-step recurrence

$$[\boldsymbol{w}_m] = a_m \frac{x_0}{2} \left\{ \begin{bmatrix} \boldsymbol{w}_{m-1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{w}_{m-1} \end{bmatrix} \right\} - b_m \begin{bmatrix} 0 \\ \boldsymbol{w}_{m-2} \\ 0 \end{bmatrix}. \tag{1}$$

The double-step recurrence approach in USWinR2 is the fastest, while USWinR1A is a good compromise between speed and program complexity.

The routines above calculate the windows without the need to evaluate trigonometric functions. They are drop-in compatible replacements for USWin with considerably reduced computation complexity with as good, or better accuracy compared to USWin.

Other routines in the extras folder.

- USWinSeries – Ultraspherical window calculated using a power series formulation.
- Cheby1Poly – Evaluate a closed-form expression for a Chebyshev polynomial of the first kind ($\alpha = 0$).
- Cheby2Poly – Evaluate a closed-form expression for a Chebyshev polynomial of the second kind ($\alpha = 1$).
- USPolyCoef – Find the coefficients of a direct-form polynomial corresponding to an ultraspherical polynomial with parameters N and α.

Sampled Continuous-Time Ultraspherical Windows

A CT ultraspherical window with N coefficients is characterized by two parameters β and α, where β determines the trade-off between the width of the main lobe and the sidelobe attenuation. A sampled

CT ultraspherical window can be a very good approximation to a DT ultraspherical window for moderatelylarge values of N, but can be computed much faster.

- `CTUSWinN` – Find N equally-spaced samples of a CT window for a window parameterized by β, and α. This routine uses either a modified or conventional sampling pattern depending on the values of α.
- `SCTDolphChebyWin` – Find samples of a sampled CT Dolph-Chebyshev window ($\alpha = 0$). This routine optimized the values for the end points of the window.
- `FRCTUSWin` – Calculate samples of the frequency response for a CT ultraspherical window.

`extras` Folder

- `CTUSWint` - Calculate samples of the spectrum of a CT ultraspherical window at arbitrary sample points.
- `FRCTCheby1Win` – Calculate samples of the spectrum a CT Dolph-Chebyshev window ($\alpha = 0$).
- `FRCTCheby2Win` – Calculate samples of the spectrum a CT Kaiser-Bessel window ($\alpha = 1$).

Test Routines

The test folder has routines that exercise the routines for both discrete-time and continuous-time ultraspherical windows. Each test routine prints results to the screen and also to a diary file (`.log` extension). The contents of the diary file are compared to a reference diary file (`.ref` extension) and any differences are displayed.

- `tUSPolyCalc` – Test ultraspherical polynomial recurrence.
- `tUSWin` – Compare IDFT / recurrence windows.
- `tUSWinCalc` – Test alternate implementations, various N and α.
- `tUSWinR` – Test recurrence-based windows calculations for small N to verify initial conditions.
- `tUSWinTime` – Compare calculation times.
- `tCTUSWinN` – Test sampled CT windows (with modified/conventional sampling patterns).