Bandwidth Efficient Filter Banks for Transmultiplexers

by

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Abstract

This thesis addresses the problem of simultaneously transmitting several data signals across a single channel. For this purpose, a transmultiplexer that uses modulated filter banks is studied. Modulated filter banks comprise filters that are bandpass versions of a lowpass prototype. The filters serve to assign portions of the channel bandwidth to the data signals. The impulse responses of the filters are parameterized by a center frequency, delay and phase factor. The objectives in configuring modulated filter banks are to use the full channel bandwidth for transmission, cancel crosstalk between signals (arises when signals share bandwidth) and cancel intersymbol interference in each data signal. Assuming an ideal channel, a synthesis procedure is developed by assigning a bandwidth to the lowpass prototype and deriving relationships among the center frequencies, delays and phases such that the entire channel bandwidth is utilized and crosstalk is cancelled. New design procedures for an FIR lowpass prototype are proposed such that the intersymbol interference is suppressed. One design method is based on a minimax criterion. Another approach involves an unconstrained optimization of an error function.

The synthesis procedure leads to five bandwidth efficient transmultiplexers. Three of the systems implement multicarrier Quadrature Amplitude Modulation (QAM) and two accomplish multicarrier Vestigial Sideband Modulation (VSB). The performance of the five systems is compared with filters obtained by the new design approaches. Also, the issue of channel distortion is addressed. Finally, the transmultiplexers can be converted into new subband systems.

Sommaire

La transmission simultanée de plusieurs signaux numériques sur un canal unique constitue le sujet de cette thèse. Pour accomplir cette tâche, un transmultiplexeur utilisant un banc de filtres modulés est étudié. Les bancs de filtres modulés sont formées des filtres qui sont des versions passe-bandes de prototype passe-bas. Les filtres ont pour fonction l'attribution de portions de la bande de fréquence du canal aux signaux numériques. Les paramètres servant à caractériser les réponses impulsionnelles des filtres sont la fréquence centrale, le délai et le facteur de phase. Les objectifs, lors de la configuration de bancs de filtres modulés, sont l'utilisation de la pleine largeur de bande lors de la transmission, l'élimination de la diaphonie entre les signaux (ceci survient lorsque les signaux partagent une même bande de fréquence) et l'élimination de l'interférence entre symboles dans chacun des signaux numériques. Assumant un canal idéal, une méthode de synthèse est développée en assignant une bande de fréquence au prototype passe-bas et en dérivant des relations entre les fréquences centrales, les délais et les phases qui assurent une entière utilisation de la bande de fréquence du canal et l'élimination de la diaphonie. De nouvelles méthodes de conception de prototype passe-bas RIF, assurant l'élimination de l'interférence entre symboles, sont proposées. Une méthode de conception est basée sur le critère minimax. Une autre approche utilise une optimisation sans contraintes d'une fonction d'erreur.

La méthode de synthèse conduit à cinq transmultiplexeurs utilisant la bande de fréquence efficacement. Trois des systèmes accomplissent QAM, alors que deux autres utilisent VSB. La performance de ces cinq systèmes, utilisant les filtres obtenus avec les nouvelles méthodes de conception, est évaluée. De plus, le problème de la distorsion provoquée par le canal est considéré. Finalement, les transmultiplexeurs peuvent être convertis en nouvelles formes de systèmes en sous-bandes.

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Chapter 1 Introduction

This thesis addresses the problem of simultaneously transmitting several data signals across a single channel. The data signals are discrete time continuous amplitude signals. In proceeding with this problem, we study a type of multirate system [1] known as a transmultiplexer. Originally, the term transmultiplexer was referred to as a device that converts between time division multiplexed (TDM) and frequency division multiplexed (FDM) formats [2]. In this thesis, a transmultiplexer is viewed in a more general context. We refer to a transmultiplexer as a multi-input, multi-output system that uses sampling rate alteration and filtering to combine N signals for transmission across a channel and then recover the N input signals. It consists of two subsystems, namely, a transmitter and receiver as shown in Fig. 1.1. At the transmitter, the N input data signals are obtained by sampling continuous time signals at a certain rate. They are then combined into a single composite signal operating at N times the original sampling rate. Implicit modulation of the data signals is accomplished by the sampling rate increase. The filters assign a frequency band to each data signal for transmission. The composite signal is sent over a channel. At the receiver, the original data signals are separated from the composite signal by filtering and a sampling rate decrease. The data signals are recovered at the original sampling rate.

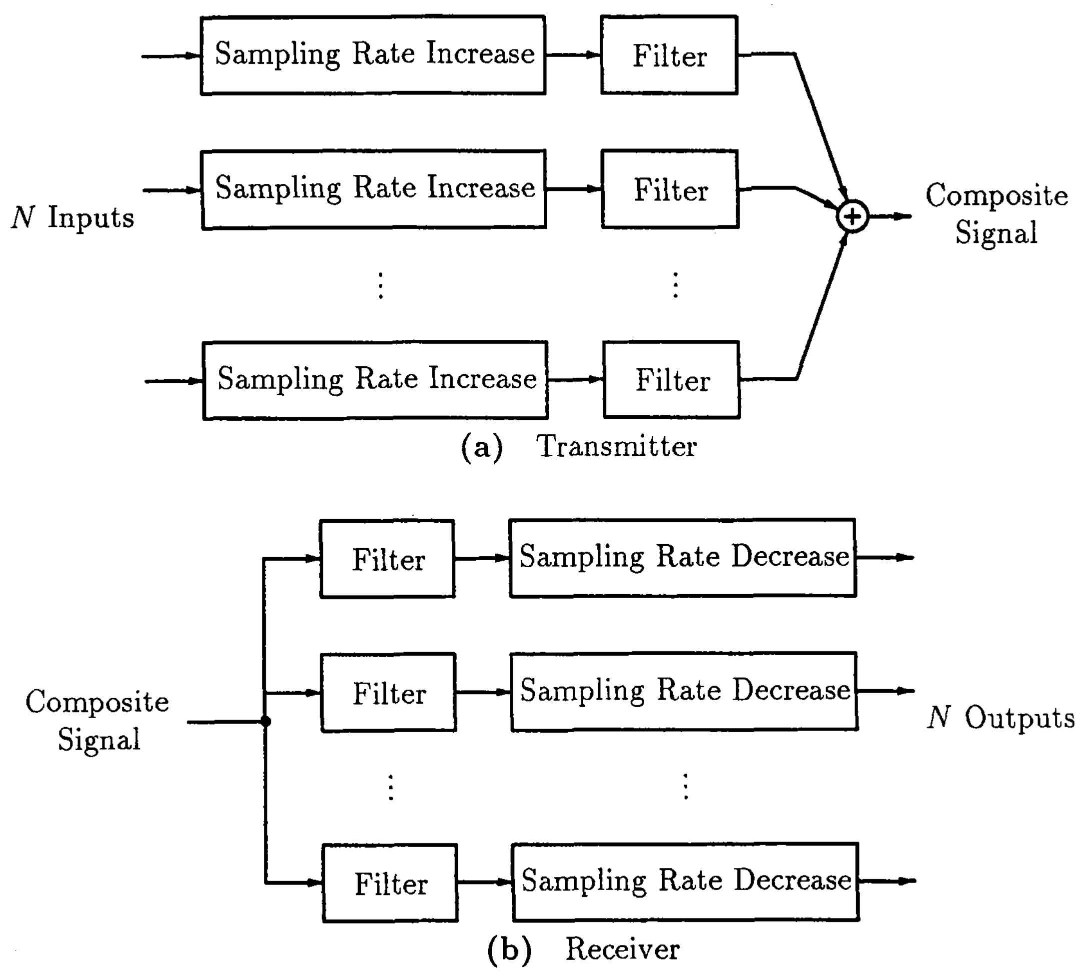


Fig. 1.1 General transmultiplexer structure

The goal in configuring transmultiplexers is to multiplex N signals at a certain sampling rate into a composite signal at N times the sampling rate (at the transmitter) and then, achieve perfect reconstruction of the inputs (at the receiver). Bandwidth efficiency (which is measured in samples/second/Hz for the type of inputs that we consider) is achieved by using the full channel bandwidth thereby leaving no gaps in the frequency bands allocated to the input signals. We consider systems that accomplish frequency division multiplexing (FDM). In these systems, the composite signal is a frequency division multiplexed form of the N data signals. The full channel bandwidth is used for transmission and equal portions of the channel bandwidth is allocated to each data signal. The various signals are confined to different frequency bands thereby leading to an implicit separation of the data signals.

An application of FDM systems is in long distance transmission over telephone and groupband lines. The resulting transmultiplexers are used in multicarrier voiceband and groupband data modems. In FDM systems, the bit rate can be maximized by appropriate information assignment to each frequency band. This is brought about by assigning more bits to the bands that are less affected by the channel characteristic. In [3], the problem of maximizing the bit rate by optimal power division among frequency bands and an optimal choice of the number of bits per data symbol subject to the constraints that the total transmitted power is fixed and the probability of error of every symbol is the same (for additive white Gaussian noise) is addressed. Results show that for channels with a sharply decreasing amplitude characteristic that approaches a null, there is much potential for achieving a high bit rate by putting more transmitter power in the bands that are unaffected by the sharply decreasing amplitude characteristic. Another aspect of FDM systems is that the channel distortion is relatively lower in each of the N bands as compared to over the entire bandwidth. Since a particular data signal is only affected by the channel distortion within its allocated frequency band, equalization can be performed in each individual frequency band as opposed to the entire frequency range. The equalizers in each band only have to deal with this relatively lower distortion.

In this thesis, we are mainly interested in developing new bandwidth efficient transmultiplexers that implement FDM. Note that the previous discussion on information assignment and equalization was meant to briefly indicate why one is interested in FDM systems. The actual details of achieving high bit rates and performing adaptive equalization is outside the scope of this study. In configuring a transmultiplexer with an FDM composite signal, consider the use of ideal bandpass filters such that their frequency responses do not overlap and such that the entire available bandwidth is used. These bandpass filters allocate different portions of the channel bandwidth to each data signal. However, such ideal bandpass filters cannot be designed in practice. This problem is circumvented by using bandpass filters whose frequency responses overlap (referred to as spectral overlap, see Fig. 1.2) such that the entire bandwidth is utilized and perfect reconstruction of the inputs results. This approach allows the data signals to share some bandwidth and yet permits reconstruction of the inputs without the use of guard bands. Guard bands are used in conventional FDM to separate the data signals but result in the wastage of useful bandwidth. Moreover, the presence of spectral overlap permits the design of practical filters.

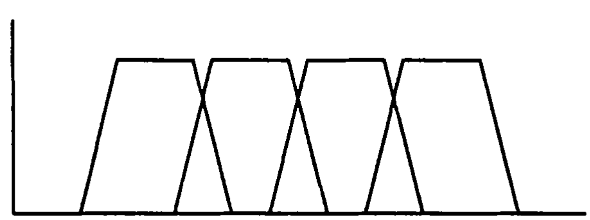


Fig. 1.2 Filter characteristics with spectral overlap

Transmultiplexers with bandpass filters having spectral overlap can be configured by different methods that lead to perfect reconstruction assuming no channel effects†. For two band systems, the standard approach is to use quadrature mirror filter (QMF) banks [4] or the Smith-Barnwell structure [5]. In the case of N bands, the use of tree-structured QMF banks [1], a matrix formalism [6][7], lossless structures [8] and modulated filter banks [9][10] accomplish perfect reconstruction.

Of the various methods that implement FDM, the focus of the research is to explore modulated filter banks in depth. Modulated filter banks have a specific structure in that all the filters are frequency shifted versions of a lowpass prototype. The filters are obtained by multiplying the lowpass impulse response by a modulating function having a specified center frequency and phase shift. This leads to a set of bandpass filters whose spectra are centered at various frequencies which are usually equally spaced. The inherent structure of modulated filter banks implies that only the design of a lowpass prototype is required to obtain complete control of the bandpass frequency responses. Also, modulated banks have been shown to lend themselves to a computationally efficient implementation through the use of a polyphase network and fast transforms [10][11].

Now, we have focused the investigation to the study of modulated filter banks. The main motivation that commences the investigation is to develop alternate configurations for modulated filter banks that accomplish perfect reconstruction. This is equivalent to examining the various ways of specifying the lowpass prototype and the parameters of the modulating function such that we get modulated filter banks

† Although these methods were originally proposed for a subband system (explained later), they are applicable to transmultiplexers.

that reconstruct the input data signals. In proceeding, we note that the modulated systems in [9][10][11] have a specific approach to describe the filters and moreover, use distinct center frequencies. We provide an additional degree of freedom in describing the filters by introducing delay factors. The resulting filters are delayed and frequency shifted versions of a lowpass prototype obtained by multiplying the lowpass impulse response by a modulating function having a specified center frequency and phase shift and then applying a delay factor. The presence of delay factors allows for the possibility of using the same center frequency to transmit two signals (a concept used in analog communication systems to send two signals in quadrature at the same frequency). The use of repeated center frequencies leads to complete spectral overlap between the corresponding bandpass filters which must be cancelled to reconstruct the inputs. Given the main aim of configuring new systems, we proceed by formulating a synthesis procedure for modulated filter banks in a transmultiplexer such that perfect reconstruction is accomplished assuming an ideal channel.

The synthesis procedure leads to the configuration of new transmultiplexers. There are two classes of systems with equally spaced center frequencies. In one type, all the center frequencies are distinct with one signal being sent at each frequency. Another type of transmultiplexer uses repeated center frequencies. Two signals are sent in quadrature at each repeated center frequency. Some of the contributions of the work lie in the formulation of the synthesis procedure, configuration of the new systems and their interpretation from a communications point of view [12][13]. Other contributions include new design procedures for a finite impulse response (FIR) lowpass prototype to be used in the new transmultiplexers [14][15][16] and a performance

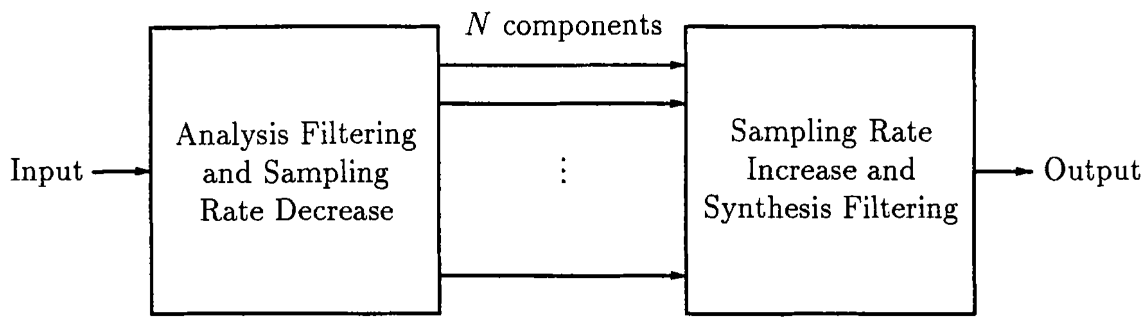


Fig. 1.3 General illustration of a subband system

evaluation of the systems [14][15]. Based upon an analysis of the new systems, the design procedures take the practical degradations into account in forming an optimal prototype. The investigation also provides some insight into the complementary nature of transmultiplexers and subband systems. Finally, the issue of channel distortion in transmultiplexers is discussed [17].

Note that the synthesized transmultiplexers can be converted into new subband systems due to the complementary nature of the two systems. A subband system is a single-input, single-output multirate system that is commonly used in medium bit rate speech coding. A general block diagram is given in Fig. 1.3. The input is split into N components by a set of analysis bandpass filters. These N components are then converted to a lower sampling rate. For speech coding applications, these components are coded in accordance with their perceptual significance. A set of synthesis filters acting on the N components results in the input being recovered at the original sampling rate. The approaches based on QMF banks [4], the Smith-Barnwell structure [5], tree-structured QMF banks [1], a matrix formalism [6][7], lossless structures [8] and modulated filter banks [9][10][11] achieve perfect reconstruction in a subband system. In fact, these approaches were originally proposed for subband systems.

The new subband systems formed from the synthesized transmultiplexers use modulated filter banks. Also, the subband systems belong to one of two classes. The subband systems which use only distinct center frequencies split the input into N components that represent different frequency ranges. The subband systems which use repeated center frequencies are unusual. Each of the repeated center frequencies establishes signal components that exist in quadrature and represent the same frequency range.

1.1 Scope and Organization of the Thesis

The entire thesis is organized into seven chapters. After the introduction, Chapter 2 provides background material concerning the input-output descriptions of transmultiplexers and subband systems and the achievement of perfect reconstruction. The complementary nature of the two systems is also discussed. The latter part of the chapter describes the research problem and the approach used.

Chapter 3 gives the transmultiplexer synthesis procedure in detail. Then, five different crosstalk-free transmultiplexers are synthesized and described from a communications point of view. New subband systems arise as complements of the synthesized transmultiplexers. The two band case is treated in more detail.

Chapters 4 and 5 are devoted to formulating procedures to design the lowpass prototype. In Chapter 4, we consider methods based on a minimax criterion that simultaneously assure a lowpass behaviour and attempt to suppress the intersymbol interference. In Chapter 5, an optimized design method based on the minimization of an error function is described. The error function is formulated so as to take practical degradations into account. Design examples are provided in both chapters. Also, the performance of the systems is evaluated for both the minimax and optimized design approaches.

Chapter 6 provides methods to configure a channel compensation filter when channel distortion is present. The channel compensation filter cancels crosstalk in the presence of a channel but leaves residual intersymbol interference. The relative performance of these methods is discussed in terms of suppression of the intersymbol interference. Chapter 7 records the conclusions of the investigation and gives some suggestions for future research.

Chapter 2 Multirate Digital Filter Banks

This chapter discusses background material on transmultiplexers and subband systems. A mathematical description of the two systems leads to the formulation of the perfect reconstruction property. Also, this establishes the complementary nature of the two systems (a concept used later in the thesis). Methods to achieve perfect reconstruction are described. Finally, the focus of the research problem and the approach used are discussed.

2.1 Transmultiplexers and Subband Systems

2.1.1 Interpolation and Decimation

Multirate systems use both interpolation and decimation to accomplish sampling rate alteration. The basic notion of interpolation lies in filling in a set of function values between two known values. Consider a discrete time signal obtained by sampling a continuous time signal. Interpolation of this signal is a two step process. First, the insertion of $N-1$ zero-valued samples between each pair of sample values of the discrete time signal is referred to as sampling rate expansion by an integral factor

N. The resulting output discrete time signal is subsequently filtered to provide a smooth transition between the nonzero samples. This smooth transition consists of estimates of the continuous time signal between the already known nonzero samples. The filtered signal can be viewed as a representation of a more finely sampled version of the continuous time signal in which the new sampling rate is N times the original sampling rate.

The process of decimation accomplishes sampling rate reduction. Again, consider a discrete time signal obtained by sampling a continuous time signal. The extraction of every Nth sample of the discrete time signal is referred to as sampling rate compression by an integral factor N. The resulting output can be obtained from the continuous time signal at $1/N$ times the original sampling rate. Note that decimation usually includes lowpass filtering prior to sampling rate compression to avoid aliasing at the lower rate.

2.1.2 Transmultiplexer

A multi-input, multi-output transmultiplexer is shown in Fig. 2.1. At the transmitter, implicit modulation is accomplished by the sampling rate expander (symbolically denoted by $N \uparrow$) since the spectrum of the input signal is replicated with period $2\pi/N$. An implicit set of carrier frequencies at multiples of $2\pi/N$ results. The combining filter bank (comprising the combining filters $A_i(z)$) allocates different portions of the channel bandwidth to the various input signals by selecting a set of N center frequencies for the purposes of transmission. The outputs of the combining filters are multiplexed into one composite signal. The composite signal is sent over a channel. At the receiver, the composite signal is passed through a parallel structure of separation filters $B_i(z)$. The sampling rate compressors (symbolically denoted by $N\downarrow$) process each of the outputs of the separation filters to yield the resultant output signals. The separation filter bank (consisting of the separation filters) and the sampling rate compressors ensure that the resulting output signals depend only on their corresponding inputs. This eliminates the influence of other inputs (crosstalk). Note that the sampling rate expansion and compression are performed synchronously at the same rate and in phase with each other.

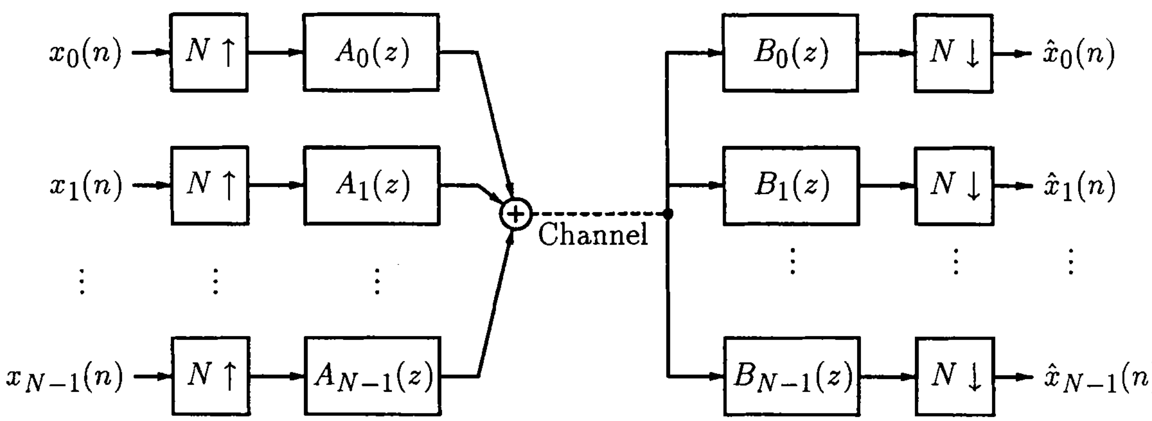


Fig. 2.1 A transmultiplexer system

Assuming that there is no channel distortion, the input-output relations are given by

$$\hat{X}_i(z) = \frac{1}{N}\sum_{k=0}^{N-1} X_k(z) \sum_{l=0}^{N-1} A_k(z^{\frac{1}{N}}W^{-l})B_i(z^{\frac{1}{N}}W^{-l}) \quad \text{for} \quad 0 \le i \le N-1\ , \qquad (2.1)$$

or equivalently (note the change from z to z^N),

$$\hat{X}_i(z^N) = \frac{1}{N}\sum_{k=0}^{N-1} X_k(z^N) \sum_{l=0}^{N-1} A_k(zW^{-l})B_i(zW^{-l}) \quad \text{for} \quad 0 \le i \le N-1\ , \qquad (2.2)$$

where $W = e^{-j\frac{2\pi}{N}}$. Each output signal $\hat{X}_i(z^N)$ is related to each input signal $X_k(z^N)$ via a transfer function $\frac{1}{N}T_{ki}(z^N)$ where

$$T_{ki}(z^N) = \sum_{l=0}^{N-1} A_k(zW^{-l})B_i(zW^{-l}) \ . \tag{2.3}$$

When $k \neq i$, $T_{ki}(z^N)$ is called a crosstalk function and represents the contribution of the undesired input $X_k(z^N)$ to the output $\hat{X}_i(z^N)$. We refer to the input-output transfer function at the ith terminal as $T_{ii}(z^N)$. For eliminating crosstalk ($T_{ki}(z^N) = 0$ for $k \neq i$) and achieving an identical input-output relation $T_{ii}(z^N) = T(z^N)$ for every terminal i, the matrix equation

$$\mathbf{A}(z)\mathbf{B}^T(z) = T(z^N)\mathbf{I} \tag{2.4}$$

must be satisfied where

$$\mathbf{A}(z) = \begin{bmatrix} A_0(z) & A_0(zW^{-1}) & \cdots & A_0(zW^{-N+1}) \\ A_1(z) & A_1(zW^{-1}) & \cdots & A_1(zW^{-N+1}) \\ \vdots & \vdots & & \vdots \\ A_{N-1}(z) & A_{N-1}(zW^{-1}) & \cdots & A_{N-1}(zW^{-N+1}) \end{bmatrix}, \tag{2.5}$$

$$\mathbf{B}(z) = \begin{bmatrix} B_0(z) & B_0(zW^{-1}) & \cdots & B_0(zW^{-N+1}) \\ B_1(z) & B_1(zW^{-1}) & \cdots & B_1(zW^{-N+1}) \\ \vdots & \vdots & & \vdots \\ B_{N-1}(z) & B_{N-1}(zW^{-1}) & \cdots & B_{N-1}(zW^{-N+1}) \end{bmatrix}, \tag{2.6}$$

and $\mathbf{I}$ is the identity matrix. If the above matrix equation is satisfied, each of the output signals $\hat{X}_i(z) = \frac{1}{N}T(z)X_i(z)$. Intersymbol interference is present if the samples at the output depend on past and future input samples. Intersymbol interference is eliminated if and only if $T(z)$ is of the form cz^{-p}. Then, perfect reconstruction is achieved in that the output samples are a scaled and delayed version of the input samples.

2.1.3 Subband System

Figure 2.2 depicts a single-input, single-output subband system. With perfect channels, the input-output description is

$$\hat{X}(z) = \frac{1}{N} \sum_{l=0}^{N-1} X(zW^{-l}) \sum_{k=0}^{N-1} A_k(zW^{-l}) B_k(z) \ . \tag{2.7}$$

The output is related to the input and its frequency shifted versions by a system function $\frac{1}{N}T_l(z)$ where $T_l(z) = \sum_{k=0}^{N-1} A_k(zW^{-l})B_k(z)$. For $l \neq 0$, we refer to $T_l(z)$ as aliasing functions. Aliasing is eliminated if $\hat{X}(z)$ is not influenced by any of the frequency shifted versions of $X(z)$. Therefore, the aliasing functions should be zero. In addition, perfect reconstruction is achieved if and only if the input-output transfer function, $T_0(z) = cz^{-p}$.

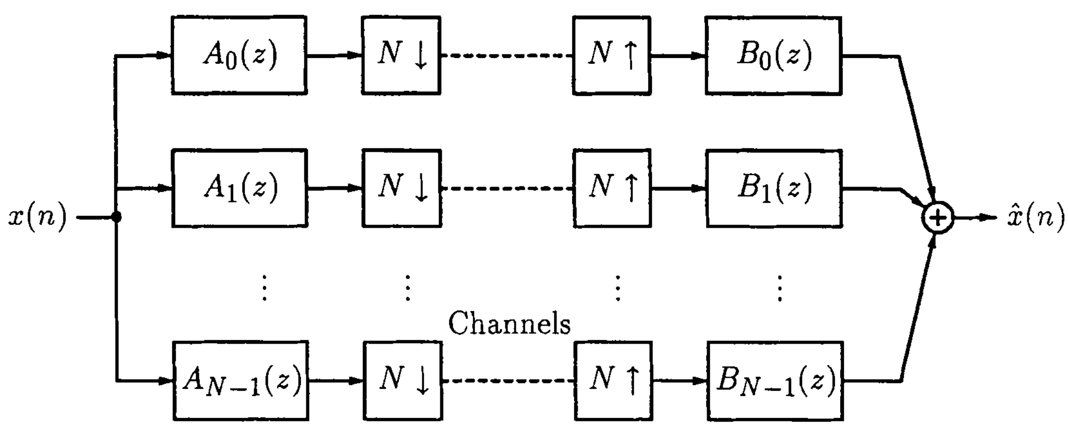


Fig. 2.2 A subband system

The cancellation of aliasing is equivalent to configuring the analysis filters $A_i(z)$ and the synthesis filters $B_i(z)$ to satisfy the system of equations $\mathbf{A}^T(z)[B_0(z) \quad B_1(z) \quad \cdots \quad B_{N-1}(z)]^T = [T_0(z) \quad 0 \quad \cdots \quad 0]^T$. This is equivalent to satisfying the matrix equation

$$\mathbf{A}^T(z)\mathbf{B}(z) = \begin{bmatrix} T_0(z) & 0 & \cdots & 0 \\ 0 & T_0(zW^{-1}) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & T_0(zW^{-N+1}) \end{bmatrix} . \tag{2.8}$$

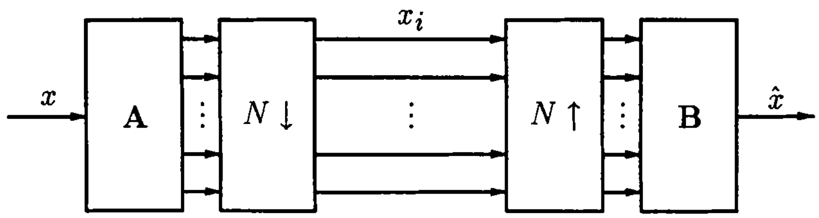
If the above matrix equation is satisfied, the output signal $\hat{X}(z) = \frac{1}{N}T_0(z)X(z)$. To provide a distinction with transmultiplexers, the filter banks in subband systems are referred to as analysis and synthesis banks.

2.1.4 Complementary Nature of the Systems

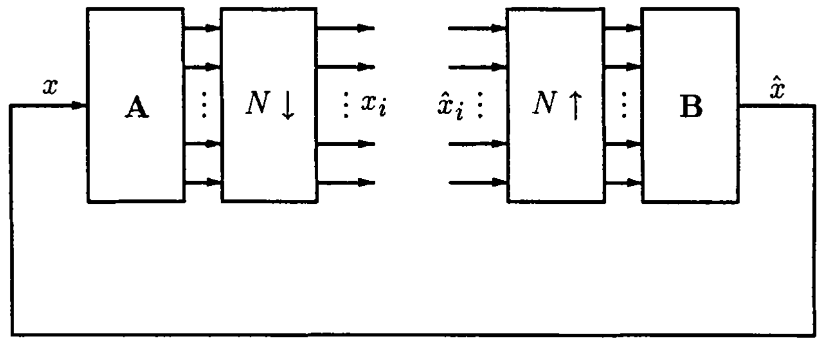
The fundamental complementary nature between transmultiplexers and subband systems relates crosstalk cancellation in the former and aliasing elimination in the latter [7]. We continue to assume that there are no channel effects in both the transmultiplexer and the subband system. It has been shown in [7] that crosstalk and aliasing cancellation are equivalent if and only if the product of the $\mathbf{A}(z)$ and $\mathbf{B}(z)$ matrices (one of them being transposed) is equal to a function in z^N multiplied by the identity matrix. By relating Eqs. (2.4) and (2.8), this is equivalent to stating that any combining/separation filter banks that eliminate crosstalk and achieve the same input-output transfer function for all pairs of corresponding terminals in a transmultiplexer will cancel aliasing when utilized as analysis/synthesis filter banks in a subband system. However, the reverse is not true unless the input-output transfer function of the subband system is a function of z^N. Analysis/synthesis filter banks for a subband system that cancel aliasing and achieve an input-output transfer function in z^N result in the relationship $\hat{X}(z) = \frac{1}{N}T(z^N)X(z)$. These same filter banks eliminate crosstalk in a transmultiplexer and achieve $\hat{X}_i(z) = \frac{1}{N}T(z)X_i(z)$ for $i = 0$ to $N-1$.

A further interpretation of this result is as follows. Suppose we design a subband system that achieves perfect reconstruction. In general, these filter banks will not cancel crosstalk in a transmultiplexer unless the transfer function of the subband system, $T(z) = cz^{-p}$ has a value of p which is a multiple of N. First suppose, $c = N$ and $p = 0$. The resulting filter banks can be applied in either a subband system or a transmultiplexer. Furthermore, there is a perfect complementary nature since the two systems are identity systems (the output samples are identical to the corresponding input samples; there is no delay factor). This is further motivated from the sequence of block diagram interpretations shown in Fig. 2.3. The identity subband system allows us to connect the input and output and break the connections between the sampling rate compressors and expanders thereby forming an equivalent transmultiplexer that is also an identity system. Note that the analysis filter bank in a subband system corresponds naturally to the separation filter bank in a transmultiplexer. Also, there exists a similar correspondence between the synthesis and combining filter banks.

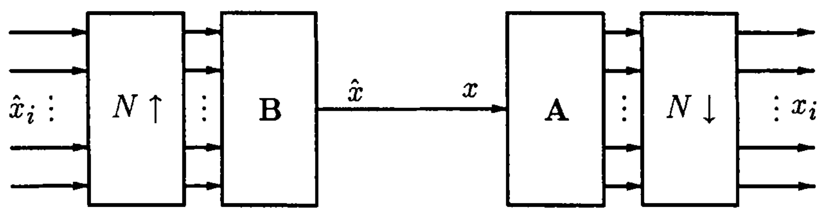
Consider the application of delay factors to an identity transmultiplexer (see Fig. 2.4(a)). The same delay factor z^{-q_1} is applied to each combining filter. Similarly, the delay factor z^{-q_2} is applied to each separation filter. The constraint $q_1 + q_2$ is a multiple of N is necessary for crosstalk cancellation to be preserved. Otherwise, the sampling rate compressors and expanders operate out of phase and crosstalk will no longer be cancelled. In addition, if $q_1 + q_2$ is a multiple of N, the delays can be moved across the sampling rate compressors and expanders without disturbing the



(a) Identity subband system



(b) Connection of output and input

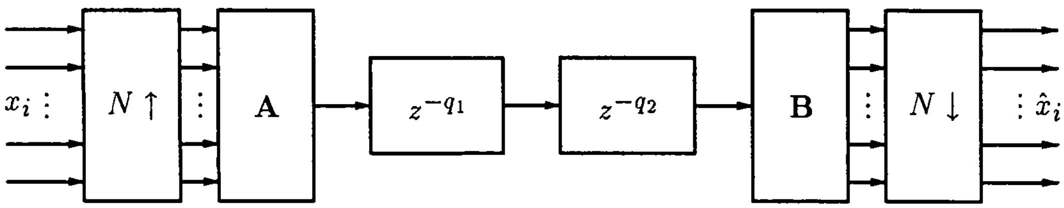


(c) Identity transmultiplexer

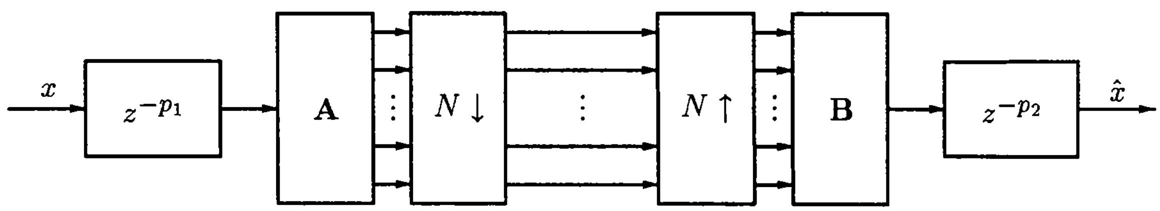
Fig. 2.3 Block diagram interpretation illustrating complementary nature

crosstalk-free nature of the system. When $q_1 + q_2$ is a multiple of N, the input-output relationship is $\hat{X}_i(z) = z^{-\frac{q_1+q_2}{N}} X_i(z)$ for $i = 0$ to $N-1$. The constraint that $q_1 + q_2$ be a multiple of N for maintaining crosstalk cancellation holds when applying delay factors to any crosstalk-free transmultiplexer.

The identity subband system can be modified by adding delay factors to the analysis and synthesis filter banks (see Fig. 2.4(b)). The same delay factor z^{-p_1} is applied to each analysis filter. Similarly, the delay factor z^{-p_2} is applied to each synthesis filter. Now, the input-output transfer function is $T(z) = Nz^{-p}$ where $p = p_1 + p_2$. Note that the alias-free nature of any subband system is preserved after applying the delay factors z^{-p_1} and z^{-p_2} to the analysis and synthesis filter banks respectively. In a practical approach, the delay factors are chosen so that causal filter banks result.



(a) Transmultiplexer



(b) Subband system

Fig. 2.4 Application of delay factors

The inherent difference between transmultiplexers and subband systems concerning the application of delay factors lies in the greater freedom that exists in choosing the delay factors for subband systems. This returns us to the principle that any crosstalk-free transmultiplexer with the same input-output transfer functions for every pair of terminals can be converted to an alias-free subband system.

2.1.5 Network Duality

Transmultiplexers and subband systems are configured by cascading two subsystems in different orders. One is a multi-input, single-output system that comprises a parallel connection of sampling rate expanders and filters. The other is a single-input, multi-output system consisting of a parallel connection of filters and sampling rate compressors. Digital network transposition transforms one subsystem into another. The process of network transposition involves interchanging the roles of the input and output, reversing the direction of all branches and replacing branch operations by their duals [1]. Since a filter is its own dual and sampling rate expansion/compression are dual operations [1], the two subsystems are transposes of each other. Furthermore, since a network and its transpose are duals, the two structures are dual systems. The two dual systems are cascaded with each other to yield two complementary multirate systems, namely, the transmultiplexer and the subband system.

By performing network transposition, we see that the duals of subband systems and transmultiplexers are again subband systems and transmultiplexers with the filter banks interchanged. Consider a subband system which is in general linear and time-varying. The dual subband system is also linear and time-varying but is described by different aliasing functions than the original system. A frequency shifted version of the aliasing function $T_l(z)$, namely, $T_l(zW^l)$, of the original system is equal to the aliasing function $T_{N-l}(z)$ of the dual system. The subband system becomes time-invariant when aliasing is cancelled and is described by an input-output transfer function $T(z)$. Therefore, the dual will also be alias-free and have the same $T(z)$ [1].

Therefore, as shown in [18], swapping the filter banks preserves aliasing cancellation and maintains the same input-output transfer function.

Now, consider a transmultiplexer which in general is not crosstalk-free. The dual transmultiplexer is also not crosstalk-free. The input-output transfer functions $T_{kk}(z)$ ($k = 0$ to $N-1$) are the same for both systems. The crosstalk functions $T_{kl}(z)$ in the original network (relating the output at terminal l to the input at terminal k) are equal to the functions $T_{lk}(z)$ of the dual network (relating the output at k to the input at l). If a transmultiplexer is crosstalk-free, the dual transmultiplexer formed by swapping the filter banks is also crosstalk-free and has the same input-output transfer functions as the original system.

The swapping property which addresses the question of whether or not exchanging the positions of the filter banks preserves the reconstruction property was discussed in [18] for subband systems. We have shown that the same property holds for a transmultiplexer with no specific assumptions about the filters or about N. Moreover, we have provided the interpretation in terms of network transposition as opposed to a direct mathematical proof. A mathematical proof starts by swapping the filter banks of a crosstalk-free transmultiplexer and examines the new matrix product

$$\begin{aligned} \mathbf{B}(z)\mathbf{A}^T(z) &= \left(\mathbf{A}(z)\mathbf{B}^T(z)\right)^T \\ &= T(z^N)\mathbf{I} \, , \end{aligned} \tag{2.9}$$

thereby establishing the swapping property†.

† The proof assumes that the input-output transfer function is the same for each pair of corresponding terminals. It can be extended to the case of having different input-output transfer functions.

2.2 Perfect Reconstruction Property

Given the requirements on $\mathbf{A}(z)$ and $\mathbf{B}(z)$ for transmultiplexers and subband systems as in Eqs. (2.4) and (2.8), methods to achieve perfect reconstruction are discussed. First, the two band case is considered. Then, we proceed to the case of arbitrary N.

2.2.1 Two Band Case

In two band systems, the classical solution is to use quadrature mirror filter banks (QMF) [4][19]. These banks consist of a lowpass/highpass filter pair whose magnitude responses are symmetric about the quarter sampling frequency $\pi/2$. A one prototype QMF system [4] is described by the following filter banks.

$$\begin{aligned} a_0(n) &= h(n) & b_0(n) &= h(n) \\ a_1(n) &= (-1)^n h(n-1) & b_1(n) &= (-1)^n h(n+1) \end{aligned} \tag{2.10}$$

For a transmultiplexer, the common input-output transfer function is

$$T(z^2) = H^2(z) + H^2(-z) \; . \tag{2.11}$$

This results in the relationship $\hat{X}_i(z) = \frac{1}{2}T(z)X_i(z)$ for $i = 0$ and 1. In the complementary subband system, $\hat{X}(z) = \frac{1}{2}T(z^2)X(z)$. To make $T(z) = cz^{-p}$ and thereby achieve perfect reconstruction, the even-indexed samples of the impulse response of $H^2(z)$ must be zero except for a reference coefficient at a time index of $2p$. The odd-indexed samples of $H^2(z)$ are arbitrary and can be used to shape the frequency response of the filter. A filter with regular zero crossings in its impulse response except for a reference coefficient is called a Nyquist filter. In this case, $H^2(z)$ is a Nyquist filter with a zero crossing interval equal to two samples.

The two band system described by Eq. (2.10) can be modified to include two prototypes $H(z)$ and $G(z)$ as follows.

$$\begin{aligned} a_0(n) &= h(n) & b_0(n) &= g(n) \\ a_1(n) &= (-1)^n g(n-1) & b_1(n) &= (-1)^n h(n+1) \end{aligned} \tag{2.12}$$

In the general case, this is not strictly a QMF bank since the magnitude responses of the lowpass/highpass pair $H(z)$ and $G(-z)$ may not be symmetrical about $\pi/2$. However, any two filters $H(z)$ and $G(z)$ such that $H(z)G(z)$ is a Nyquist filter with a zero crossing interval of two samples results in perfect reconstruction. In addition, methods to get a lowpass $H(z)$ and $G(z)$ are given in [6][7]. A special case of Eq. (2.12) arises when $G(z) = H(z^{-1})$. The resulting system, known as a Smith-Barnwell structure [5], requires a Nyquist filter $H(z)H(z^{-1})$ to achieve perfect reconstruction. A lowpass Nyquist filter must be factored into its minimum and maximum phase components.

Note that the descriptions in Eqs. (2.10) and (2.12) can lead to noncausal filters. However, given the previous discussion on the application of delay factors, we can modify any noncausal bank to make it causal such that perfect reconstruction is preserved.

2.2.2 The N Band Case

The perfect reconstruction condition for the N band case depends on the product of $\mathbf{A}(z)$ and $\mathbf{B}(z)$ (one of them being transposed). The methods proposed to configure the filter banks that are based on a matrix formalism and on lossless structures impose a specific structure on $\mathbf{A}(z)$. Then, $\mathbf{B}(z)$ is determined given $\mathbf{A}(z)$ thereby rendering a particular relationship between $B_k(z)$ and $A_k(z)$. Modulated filter banks specify $A_k(z)$ and $B_k(z)$ in terms of a lowpass prototype and a modulating function. It is the characteristics of the prototype and the modulating function that ensure perfect reconstruction.

2.2.2.1 Matrix Formalism

The use of a matrix formalism in determining the filter banks has been described in the context of a subband system in [6]. The method comprises two stages. The first stage introduces a way of directly solving for the synthesis filter bank in terms of the analysis bank such that the system described by Eq. (2.8) is satisfied. This results in the cancellation of aliasing. Given the resulting input-output transfer function, the second stage is devoted to designing the filters to get perfect reconstruction.

In the first stage, a polyphase matrix is defined as $\mathbf{P}(z) = \frac{1}{N}\mathbf{F}\mathbf{A}^T(z)$ where the entries of $\mathbf{F}$ are $F(m,n) = W^{mn}$ for $m,n = 0$ to $N-1$. Then, the entries of $\mathbf{P}(z)$ are $P(i,j) = z^{-i}A_{ij}(z^N)$ for $i,j = 0$ to $N-1$ where $A_{ij}(z^N)$ is the jth polyphase component of $A_i(z)$. As opposed to $\mathbf{A}^T(z)$, $\mathbf{P}(z)$ has the advantage of being purely real and exhibits no redundancy (in $\mathbf{A}^T(z)$, each filter coefficient appears N times). It is shown in [6] that choosing $B_k(z)$ such that $[B_0(z) \quad B_1(z) \quad \cdots \quad B_{N-1}(z)]^T = [1 \quad 1 \quad \cdots \quad 1]\mathbf{C}^T(z)$ where $\mathbf{C}(z)$ is the cofactor matrix of $\mathbf{P}(z)$ results in an alias-free subband system with $\hat{X}(z) = [\mathrm{Det}\ \mathbf{P}(z)]X(z)$. The abbreviation Det refers to determinant.

At the second stage, the analysis filters are designed to reduce Det $\mathbf{P}(z)$ to the form cz^{-p}. Therefore, a specific restriction on $\mathbf{A}(z)$ is imposed to ensure that Det $\mathbf{P}(z) = cz^{-p}$. A method to design FIR filters of equal length L to satisfy the determinant constraint is discussed in [6][7]. A total of $N-1$ of the analysis filters $A_k(z)$ are each designed separately with a length L that is sufficient to get an acceptable frequency response. Also, $N-1$ of the coefficients of the remaining filter are chosen thereby leaving $L-N+1$ unknown coefficients. Note that there are $L-N+1$ nonzero coefficients in Det $\mathbf{P}(z)$. Therefore, a linear system of equations that solve for the $L-N+1$ coefficients of the remaining filter results such that Det $\mathbf{P}(z)$ is reduced to the form cz^{-p}. Note that the constraint $L > N-1$ is necessary to ensure that the determinant of $\mathbf{C}(z)$ is not zero. After designing the analysis filters, $B_k(z)$ is determined as described above.

Although perfect reconstruction is accomplished by this method, there is no direct control of the frequency response of one of the filters. Moreover, the filters $B_k(z)$ are generally longer than the $A_k(z)$ [6]. This approach based on a matrix formalism is applicable to the configuration of perfect reconstruction transmultiplexers. The combining filters $A_k(z)$ and the separation filters $B_k(z)$ can be obtained as described above. However, delay factors may have to be applied to the separation filters to achieve perfect reconstruction in a transmultiplexer.

2.2.2.2 Lossless Structures

A matrix function $\mathbf{G}(z)$ is said to be lossless [8] if it is stable and satisfies the relation

$$\mathbf{G}^{H}(z^{-1})\mathbf{G}(z) = \mathbf{I} \ , \tag{2.13}$$

where the superscript H denotes the complex conjugation of the coefficients of each entry of the matrix followed by transposition and $\mathbf{I}$ is the identity matrix. In particular, this means that $\mathbf{G}$ is unitary on the unit circle $z = e^{j\omega}$. It is known that the scattering matrix of any lossless multiport analog network is unitary [20]. Hence, the term lossless has been used due to describe any $\mathbf{G}(z)$ which satisfies Eq. (2.13) and is hence, unitary on the unit circle. In the case of a scalar function, $G(z)$ is lossless if it is stable and allpass.

In [8], the lossless property is imposed on $\mathbf{A}(z)$ in order to get a set of synthesis filters $B_k(z) = cz^{-p}A_k(z^{-1})$ for a perfect reconstruction subband system. It can be shown that by making $\mathbf{A}(z)$ lossless, a set of separation filters given by $B_k(z) = cz^{-mN}A_k(z^{-1})$ results in a perfect reconstruction transmultiplexer. A design procedure in [8] leads to a set of FIR bandpass filters $A_k(z)$ such that $\mathbf{A}(z)$ is lossless. First, the filters $A_k(z)$ are derived from a cascade of lossless building blocks composed of the product of a unitary matrix and a diagonal matrix whose entries are delay elements. The entries of the unitary matrices are jointly optimized to yield a set of bandpass filters $A_k(z)$. By examining the simple relationship between $B_k(z)$ and $A_k(z)$, we observe that their magnitude responses are identical. Moreover, the number of coefficients of the FIR $B_k(z)$ is the same as that of the FIR $A_k(z)$.

2.2.2.3 Modulated Filter Banks

In modulated filter banks, all the filters are frequency shifted versions of a lowpass prototype. This gives a set of bandpass filters whose impulse responses are of the form $h(n)\cos(\omega n + \gamma)$ where $h(n)$ is a lowpass prototype. The modulating function is described by a center frequency ω and a phase factor γ. For the case of distinct center frequencies, the prototype is bandlimited such that there is spectral overlap only between adjacent bandpass filters. Hence, any output signal at terminal l in a transmultiplexer will experience crosstalk only from input signals at adjacent terminals $l-1$ and $l+1$. The other crosstalk functions are zero since the magnitude responses of the corresponding bandpass filters are nonoverlapping. In a subband system, the only aliasing terms are those due to spectral overlap. The other aliasing terms are zero due to the bandlimitedness of the lowpass prototype. The crosstalk and aliasing terms due to spectral overlap are cancelled by fixing the parameters of the modulating function. This gives crosstalk-free transmultiplexers and alias-free subband systems with bandlimited filters. Finally, perfect reconstruction is achieved by satisfying the Nyquist criterion for zero intersymbol interference. In a practical situation, the lowpass prototype is designed to have a sufficiently high stopband attenuation and such that a Nyquist response is either approximately or exactly achieved. Modulated filter banks have the advantages of allowing for complete control of the frequency responses of the bandpass filters through the design of a lowpass prototype and being computationally efficient to implement.

The modulated filter banks in [9][10][11] were originally proposed for a subband system. The filter banks in [9][10] are applicable in a transmultiplexer. The system in [9] is not a regular structure in that the center frequencies are not equally spaced and two prototypes of different bandwidths are used. The system in [10] uses one prototype $h(n)$ which is bandlimited to no more than π/N. Also, the center frequencies are odd multiples of $\pi/2N$. Therefore, the center frequencies are equally spaced and exactly the same bandwidth is allocated to each input signal.

2.3 Focus of Research Problem

The investigation concentrates on modulated filter banks in a transmultiplexer. The main purpose is to find alternative configurations of modulated filter banks to those already described in the literature. This goal is achieved through the formulation of a synthesis procedure. The synthesis procedure allows for a systematic development in finding modulated filter banks. We start with a set of assumptions that form a characterization of the filter banks. These assumptions allow for more generality in describing the filters than in previously configured systems. Then, specific relationships among the parameters of the filters are derived such that crosstalk is cancelled and the input-output transfer function between every pair of corresponding terminals is the same. This constructive approach results in the configuration of new crosstalk-free transmultiplexers. The intersymbol interference is suppressed by designing the lowpass prototype.

The general nature of the starting assumptions provides greater flexibility in specifying the filter banks as compared to the existing systems. In particular, the assumptions made are as follows:

1. The filter banks consist of a set of bandpass filters that are modulated versions of a lowpass prototype.

2. The impulse responses of the filters are described by the impulse response of the prototype and three free parameters, namely, a center frequency, phase factor and delay.
3. Equally spaced center frequencies are used. In one case, all the frequencies are distinct. In another case, the center frequencies are allowed to repeat such that the same center frequency is used for two bands.

Note that a perfect channel is assumed. A discussion of channel distortion is given in Chapter 6.

Assumption 2 provides an extra free parameter, namely, a delay factor in describing the impulse responses of the bandpass filters as compared to existing systems that only allow for a center frequency and phase factor. The idea of permitting center frequencies to repeat allows for two signals to be sent at the same frequency as compared to existing schemes in which all the center frequencies are distinct. Additional freedom is provided over the existing N band modulated banks that have the multirate structure of Fig. 2.1, use one lowpass prototype to derive a set of bandpass filters and maintain equally spaced center frequencies.

The central objective of formulating a synthesis procedure involves the following steps.

1. The bandwidth of the lowpass prototype is determined such that (1) spectral overlap occurs only between bandpass filters centered at adjacent center frequencies and at the same center frequency and (2) the set of bandpass filters fill up the entire frequency range (0 to π).
2. Relationships among the three free parameters (center frequencies, phase factors and delays) are derived such that the resulting transmultiplexers have the following properties.
   (a) The input-output transfer function is the same for every pair of corresponding terminals.
   (b) The crosstalk components in the output data signal that arise from other data signals due to the sharing of bandwidth are eliminated.

The synthesis procedure is developed based on a bandlimited lowpass prototype. A filter $H(z)$ is a bandlimited lowpass prototype if $H(e^{j\omega})$ is exactly equal to zero in the stopband region $\omega_s \leq \omega \leq \pi$. The frequency characteristic of a general bandlimited lowpass prototype with a tapered transition band is shown in Fig. 2.5. In Step 1, we determine the stopband edge ω_s (thereby determining the bandwidth of the prototype) for the purposes of restricting spectral overlap and allowing for full bandwidth utilization. For systems in which all the center frequencies are distinct, an output signal at a particular terminal will experience crosstalk from input signals transmitted at adjacent center frequencies. For systems with repeated frequencies, there is (1) partial spectral overlap between bandpass filters centered at adjacent center frequencies and (2) complete spectral overlap between bandpass filters centered at the same center frequency. Then, an output signal at a particular terminal will experience crosstalk from input signals transmitted at adjacent center frequencies and another signal sent at the same center frequency.

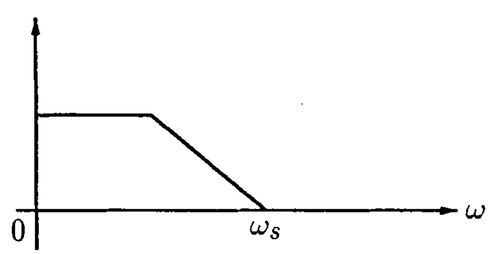


Fig. 2.5 Frequency characteristic of a general bandlimited lowpass prototype

Step 2 consists of two parts each devoted to forming relationships among the center frequencies, phase factors and delays. First, the transfer function between each pair of corresponding terminals is made to be the same. The transfer function is brought to a form which allows us to design a lowpass prototype such that the intersymbol interference is suppressed (discussed in later chapters). Also, the transmultiplexers can be converted into subband systems which split the entire spectrum of the input signal into N frequency bands. In Step 2(b), the crosstalk components due to spectral overlap are cancelled. The crosstalk between signals that do not share any bandwidth is zero for bandlimited filters.

The next chapter gives the synthesis procedure in detail. Since bandlimited filters (stopband response is exactly zero) cannot be designed, a natural question concerns how the design of a practical lowpass prototype can be performed. A practical lowpass prototype is distinguished from a bandlimited prototype in that the frequency response of the practical filter only approximates the characteristic shown in Fig. 2.5. In particular, the practical prototype has a stopband response which is small but not exactly zero (stopband attenuation is high but not infinite). In Chapters 4 and 5, new design methods for a practical FIR lowpass prototype are developed with the aim of suppressing both intersymbol interference and crosstalk.

Chapter 3 Transmultiplexer Synthesis

This chapter discusses the synthesis procedure for modulated banks in a transmultiplexer. The first step is to state the general assumptions. This includes the specification of the impulse responses of the combining and separation filters in terms of a lowpass prototype, center frequency, phase factor and delay. The synthesis procedure starts by imposing a bandwidth constraint on the lowpass prototype. Then, the input-output transfer function and the crosstalk functions are examined. This leads to new crosstalk-free transmultiplexers. The last portion of this chapter exclusively deals with two band transmultiplexers. Finally, the complementary subband systems that emerge from the synthesized transmultiplexers are discussed.

3.1 Filter Specification

In developing a synthesis procedure, the first assumption characterizes the filter banks. We confine all the filters to be modulated and delayed versions of one bandlimited lowpass prototype $h(n)$. This condition will be relaxed later to allow for two prototypes. The impulse responses of the combining filters $A_k(z)$ and the separation filters $B_k(z)$ are parameterized by a center frequency (ω_k), phase factor (α_k or β_k)

and delay (n_k or p_k). Their impulse responses are given by

$$a_k(n) = h(n - n_k) \cos\left[\omega_k(n - n_k) + \alpha_k\right] \tag{3.1}$$

and

$$b_k(n) = h(n + p_k) \cos\left[\omega_k(n + p_k) + \beta_k\right] \tag{3.2}$$

respectively†. In the z-transform domain, $A_k(z)$ and $B_k(z)$ are given by

$$A_k(z) = \frac{1}{2} z^{-n_k}[e^{j\alpha_k} H(e^{-j\omega_k} z) + e^{-j\alpha_k} H(e^{j\omega_k} z)] \tag{3.3}$$

and

$$B_k(z) = \frac{1}{2} z^{p_k}[e^{j\beta_k} H(e^{-j\omega_k} z) + e^{-j\beta_k} H(e^{j\omega_k} z)] \ . \tag{3.4}$$

The transmultiplexers have N bands. Also, N is the sampling rate expansion/compression factor.

We further assume that the center frequencies ω_k are equally spaced and lie between 0 and π (inclusive). In addition, two types of systems are considered. In one type, all the center frequencies are distinct. In the other case, center frequencies are repeated (with the exception of 0 and π) in that the same frequency is used for two bands. Finally, note that the synthesis procedure is developed given that no channel distortion is present.

3.2 Bandwidth Constraints

The first step in the synthesis procedure is to impose a bandwidth constraint on the lowpass prototype. Consider the type of system in which all the center frequencies

† Depending on the signs of n_k and p_k, either a delay or advance is used. In the remainder of the thesis, we refer to n_k and p_k as delay factors regardless of whether they are positive or negative.

are distinct. The bandwidth of the bandlimited lowpass prototype $h(n)$ (stopband response is exactly zero) is selected such that spectral overlap exists only between filters centered at adjacent center frequencies. In addition, the entire range 0 to π is utilized. Given $h(n)$, there are N bandpass filter responses centered at different frequencies and having the same bandwidth. The minimum bandwidth of the N bandpass filters such that their frequency responses are mutually exclusive (no spectral overlap), an equal bandwidth is allocated to each input and the full frequency range 0 to π is covered is π/N. Moreover, the center frequencies are odd multiples of $\pi/2N$. This translates to a minimum bandwidth of $\pi/2N$ for $h(n)$. Spectral overlap is restricted to bandpass filters centered at adjacent frequencies by allowing the lowpass prototype to have a bandwidth of no more than 100 percent in excess of its minimum bandwidth. The stopband of the bandlimited lowpass prototype $h(n)$ extends from ω_s to π where $\pi/2N \leq \omega_s \leq \pi/N$.

Now, consider the type of system in which the center frequencies repeat. Two signals are transmitted at every repeating center frequency (0 and π excluded). The minimum bandwidth of the bandpass filters which allows for filters centered at different frequencies to have mutually exclusive frequency responses is $2\pi/N$. This translates to a minimum bandwidth of π/N for $h(n)$. Moreover, there are two possible sets of center frequencies. In one set, two of the center frequencies are 0 and π and the other repeating frequencies are multiples of $2\pi/N$. Another possibility is to have all the frequencies repeat and be odd multiples of π/N†. The idea is to allow for

† We have implicitly considered the case when N is even. When N is odd, one of the center frequencies is 0 or π with the remaining center frequencies repeating. The spacing between adjacent frequencies is $2\pi/N$. The minimum bandwidth of the filters is the same as for N even.

spectral overlap only between filters centered at the same frequency and at adjacent frequencies. For both sets of center frequencies, this is possible if the lowpass prototype $h(n)$ is bandlimited to no more than 100 percent over the minimum bandwidth. The stopband of $h(n)$ extends from ω_s to π where $\pi/N \leq \omega_s \leq 2\pi/N$.

The bandwidth constraint is different for repeated and distinct center frequencies. Given the above constraints on ω_s, the development of the synthesis procedure evolves by assuming that the lowpass prototype $h(n)$ has a stopband response that is exactly zero (bandlimited prototype). Later, we will consider systems with practical filters.

We have established three sets of equally spaced center frequencies. For the case of repeated center frequencies, the two sets are

$$\text{Set 1}: \ 0, \frac{2\pi}{N}, \frac{2\pi}{N}, \frac{4\pi}{N}, \frac{4\pi}{N}, \cdots, \pi - \frac{2\pi}{N}, \pi - \frac{2\pi}{N}, \pi$$

and

$$\text{Set 2}: \ \frac{\pi}{N}, \frac{\pi}{N}, \frac{3\pi}{N}, \frac{3\pi}{N}, \cdots, \pi - \frac{\pi}{N}, \pi - \frac{\pi}{N} \, .$$

Both Sets 1 and 2 ensure complete bandwidth utilization (frequency range 0 to π is covered) given a lowpass prototype with a stopband frequency $\omega_s \geq \pi/N$. Also, spectral overlap is restricted to filters centered at the same frequency and at adjacent center frequencies if $\omega_s \leq 2\pi/N$. Note that for Sets 1 and 2, it is assumed that N is even. Later, we will see that this is necessary for realizing integral delay factors.

The set of N distinct equally spaced center frequencies is given by

$$\text{Set 3}: \ \frac{\pi}{2N}, \frac{3\pi}{2N}, \frac{5\pi}{2N}, \frac{7\pi}{2N}, \cdots, \pi - \frac{\pi}{2N} \, .$$

The center frequencies of Set 3 are the same as those in [10]. Complete bandwidth utilization is achieved given a lowpass prototype with a stopband edge $\omega_s \geq \pi/2N$.

Also, spectral overlap is restricted to bandpass filters centered at adjacent frequencies if $\omega_s \leq \pi/N$.

3.3 Input-Output Transfer Function

The next step is to make the input-output transfer function the same for every pair of corresponding terminals. The kth input-output terminal pair has a transfer function given by

$$\begin{aligned} T_{kk}(z^N) &= \sum_{i=0}^{N-1} A_k(zW^{-i})B_k(zW^{-i}) \\ &= \frac{1}{4}z^{-(n_k-p_k)}\sum_{i=0}^{N-1} W^{i(n_k-p_k)}\Big[e^{j(\alpha_k+\beta_k)}H^2(e^{-j\omega_k}zW^{-i}) \\ &\qquad + e^{-j(\alpha_k+\beta_k)}H^2(e^{j\omega_k}zW^{-i}) \\ &\qquad + 2\cos(\alpha_k-\beta_k)H(e^{-j\omega_k}zW^{-i})H(e^{j\omega_k}zW^{-i})\Big]. \end{aligned} \tag{3.5}$$

The strategy will be to try to make the transfer function $T_{kk}(z^N)$ independent of k. To this end, it is assumed that $n_k - p_k = s$ for every k. The expression for the input-output transfer function consists of three terms. Note that the last term in Eq. (3.5) will be zero for center frequencies sufficiently away from 0 and π (the spectra in the $H(\cdot)$ terms do not overlap). Specifically, this will be true for $\omega_b \leq \omega_k \leq \pi - \omega_b$ where ω_b is the maximum bandwidth of the lowpass prototype (π/N for distinct center frequencies and $2\pi/N$ for repeated center frequencies). For the center frequencies near 0 or π, choosing $\alpha_k - \beta_k$ to be an odd multiple of $\pi/2$ will suffice to set the last term to zero. We now formulate two sets of conditions for identical input-output transfer functions.

Difference Criterion

For the difference criterion, the difference between any two center frequencies is constrained to be a multiple of $2\pi/N$. We first note that the frequency response of $T_{kk}(z^N)$ is periodic in $2\pi/N$. Equation (3.5) remains unchanged if, in its first two terms, ω_k is replaced by $\omega_l = \omega_k + 2m\pi/N$ (where m is an integer) and $n_k - p_k = s$ is a multiple of N (recall that the last term is zero from the preceding discussion). Then, the same transfer functions at terminals k and l are achieved by adhering to the following set of rules.

1. If a particular ω_k does not satisfy the inequality $\omega_b \leq \omega_k \leq \pi - \omega_b$, then $\alpha_k - \beta_k$ must be an odd multiple of $\pi/2$. The same restriction holds for terminal l.
2. The phases are chosen such that $\alpha_k + \beta_k = \alpha_l + \beta_l$.
3. The delay factors are chosen such that $n_k - p_k = n_l - p_l$. Moreover, both $n_k - p_k$ and $n_l - p_l$ are multiples of N.

The above rules generate a reduced form of $T_{kk}(z^N) = T_{ll}(z^N)$ as given by

$$T_{kk}(z^N) = \frac{1}{4} z^{-(n_k - p_k)} \sum_{i=0}^{N-1} \left[e^{j(\alpha_k + \beta_k)} H^2(e^{-j\omega_k} z W^{-i}) + e^{-j(\alpha_k + \beta_k)} H^2(e^{j\omega_k} z W^{-i}) \right]. \tag{3.6}$$

Sum Criterion

It can be shown that if we confine the sum of the center frequencies $\omega_k + \omega_l = 2m\pi/N$ (where m is an integer), another set of rules for which $T_{kk}(z^N) = T_{ll}(z^N)$ emerges as follows.

1. If a particular ω_k does not satisfy the inequality $\omega_b \leq \omega_k \leq \pi - \omega_b$, then $\alpha_k - \beta_k$ must be an odd multiple of $\pi/2$. The same restriction holds for terminal l.
2. The phases are chosen such that $\alpha_k + \beta_k = -(\alpha_l + \beta_l)$.

3. The delay factors are chosen such that $n_k - p_k = n_l - p_l$. Moreover, both $n_k - p_k$ and $n_l - p_l$ are multiples of N.

This generates a reduced form for the input-output transfer function as above.

Center Frequencies

The center frequencies of Set 1 and Set 2 satisfy both the difference and sum criteria. In fact, the conditions for the two criteria are equivalent for the frequencies of Sets 1 and 2. Any two center frequencies of Set 3 satisfy either the difference or the sum criterion. At this stage, we confine $\alpha_k + \beta_k$ to be a multiple of π for Sets 1, 2 and 3. Appendix A elaborates on this aspect and justifies this choice. For the end center frequencies (those that do not satisfy the inequality $\omega_b \leq \omega_k \leq \pi - \omega_b$), the phase difference $\alpha_k - \beta_k$ is constrained to be an odd multiple of $\pi/2$. Combining this with the constraint on $\alpha_k + \beta_k$ gives the condition that the phases α_k and β_k are of the form $(2r+1)\pi/4$, where r is an integer, for the end frequencies. The end frequencies are 0 and π for Set 1, π/N and $\pi - \pi/N$ for Set 2 and $\pi/2N$ and $\pi - \pi/2N$ for Set 3.

3.4 Analysis of Crosstalk

This section analyzes the crosstalk functions for signals sent at adjacent center frequencies and the crosstalk functions for signals sent at the same center frequency. The crosstalk functions associated with signals whose allocated bandwidths do not overlap are equal to zero. We will adhere to the restrictions generated in Section 3.3 for the input-output transfer function and formulate additional conditions for cancelling crosstalk. The case when the center frequencies repeat and the case when they are distinct are considered separately. To start, we express the general crosstalk function for signals transmitted at two center frequencies ω_k and ω_l as

$$\begin{aligned} T_{kl}(z^N) &= \sum_{i=0}^{N-1} A_k(zW^{-i})B_l(zW^{-i}) \\ &= \frac{1}{4}z^{-(n_k-p_l)}\sum_{i=0}^{N-1} W^{i(n_k-p_l)}\Big[e^{j(\alpha_k+\beta_l)}H(e^{-j\omega_k}zW^{-i})H(e^{-j\omega_l}zW^{-i}) \\ &\qquad + e^{-j(\alpha_k+\beta_l)}H(e^{j\omega_k}zW^{-i})H(e^{j\omega_l}zW^{-i}) \\ &\qquad + e^{j(\alpha_k-\beta_l)}H(e^{-j\omega_k}zW^{-i})H(e^{j\omega_l}zW^{-i}) \\ &\qquad + e^{-j(\alpha_k-\beta_l)}H(e^{j\omega_k}zW^{-i})H(e^{-j\omega_l}zW^{-i})\Big] . \end{aligned} \tag{3.7}$$

The *crosstalk function* $T_{kl}(z^N)$ represents the contribution of the input $X_k(z^N)$ (transmitted at ω_k) to the output $\hat{X}_l(z^N)$. In the sequel, the four terms of which $T_{kl}(z^N)$ is comprised of are referred to as *crosstalk terms.*

3.4.1 Crosstalk: Different Center Frequencies of Sets 1 and 2

Consider the case of center frequencies belonging to Sets 1 and 2. These frequencies are multiples of π/N. For now, it is assumed that the different positive frequencies ω_k and ω_l are in the closed interval $[2\pi/N, \pi - 2\pi/N]$. Two adjacent center frequencies ω_k and ω_l are related by $\omega_l - \omega_k = 2m\pi/N$ where $m = \pm 1$. Given two adjacent frequencies, the last two crosstalk terms of Eq. (3.7) are zero due to the bandlimitedness of $H(z)$. By substituting the relationship $\omega_l - \omega_k = 2m\pi/N$ ($m = \pm 1$) in the first two terms of Eq. (3.7), noting that $e^{j\omega_k} = W^p$ where p is a multiple of 1/2 and performing algebraic manipulation to give identical crossterms in

$H(\cdot)$, we get a simplified expression for the crosstalk function as

$$T_{kl}(z^N) = \frac{1}{4} z^{-(n_k - p_l)} \sum_{i=0}^{N-1} \Big[[W^{(m-2p)(n_k-p_l)} e^{j(\alpha_k+\beta_l)} + e^{-j(\alpha_k+\beta_l)}] \\ W^{i(n_k-p_l)} H(zW^{-i+p}) H(zW^{-i-m+p}) \Big] \; . \tag{3.8}$$

From Eq. (3.8), we develop a general rule relating the phases, delays, m and p as given by (discussion in Appendix B)

$$\alpha_k + \beta_l = \pi \left[\frac{(m-2p)(n_k - p_l)}{N} + \frac{1}{2} \right] \; . \tag{3.9}$$

Since $m = \pm 1$, we have considered crosstalk due to spectral overlap between signals transmitted at any two adjacent center frequencies in the closed interval $[2\pi/N, \pi - 2\pi/N]$. Then, Eq. (3.9) becomes

$$\alpha_k + \beta_l = \pi \left[\frac{(\pm 1 - 2p)(n_k - p_l)}{N} + \frac{1}{2} \right] \; . \tag{3.10}$$

In seeking solutions to Eq. (3.10), we first note that p is either an even or odd multiple of $1/2$ thereby making $\pm 1 - 2p$ odd or even respectively. Equation (3.10) depicts a general relationship between two unknowns $\alpha_k + \beta_l$ and $n_k - p_l$. In establishing particular relationships between these two unknowns, we express $n_k - p_l$ as a rational multiple of N, namely aN/b where a and b are relatively prime. To realize integral delay factors, aN/b must be an integer thereby imposing a restriction on N or the number of bands to be an integral multiple of b. To avoid excessive restrictions on N, b must be kept to a minimum. We consider the cases in which $b = 1$ (no restriction on N) and $b = 2$ (N is even). This gives two different types of solutions to Eq. (3.10) which are necessary since two signals are sent with the same center frequency. Also, N is constrained to be even as a result.

3.4.1.1 Set 1

In Set 1, p is an even multiple of $1/2$ (center frequencies are even multiples of π/N). The two types of solutions to Eq. (3.10) are given below.

Solution One

1. The delays are chosen such that $n_k - p_l$ is a multiple of N.
2. The phases are chosen such that $\alpha_k + \beta_l$ is an odd multiple of $\pi/2$.

Solution Two

1. The delays are chosen such that $n_k - p_l$ is an odd multiple of $N/2$.
2. The phases are chosen such that $\alpha_k + \beta_l$ is a multiple of π.

The only remaining crosstalk due to spectral overlap occurs between the end center frequency $\omega_k = 0$ and $\omega_l = 2\pi/N$. Retaining the restriction on α_k and β_k for the end center frequencies and the difference in the delay factors to be as above, two ways of eliminating crosstalk are as follows.

1. The delays are chosen such that $n_k - p_l$ and $n_l - p_k$ are multiples of N. The phases α_k and β_k are either $\pm\pi/4$ or $\pm 3\pi/4$. The phases α_l and β_l are odd multiples of $\pi/2$.
2. The delays are chosen such that $n_k - p_l$ and $n_l - p_k$ are odd multiples of $N/2$. The phases α_k and β_k are either $\pm\pi/4$ or $\pm 3\pi/4$. The phases α_l and β_l are multiples of π.

The same techniques result in cancelling crosstalk between signals sent at the other center frequencies of $\pi - 2\pi/N$ and π.

3.4.1.2 Set 2

For Set 2, p is an odd multiple of $1/2$ (center frequencies are odd multiples of π/N). A type of solution to Eq. (3.10) is given below.

Solution

1. The delays are chosen such that $n_k - p_l$ is a multiple of $N/2$.
2. The phases are chosen such that $\alpha_k + \beta_l$ is an odd multiple of $\pi/2$.

For the end center frequency $\omega_k = \pi/N$, spectral overlap occurs with $\omega_l = 3\pi/N$. By examining the crosstalk function, it is found that the elimination of crosstalk is feasible if both of the conditions below are satisfied.

1. The delays are chosen such that $n_k - p_l$ and $n_l - p_k$ are multiples of $N/2$.
2. The phases are chosen such that (α_k, β_l) and (β_k, α_l) are $(\pi/4, \pi/4 \pm m\pi)$, $(-\pi/4, -\pi/4 \pm m\pi)$, $(3\pi/4, 3\pi/4 \pm m\pi)$ or $(-3\pi/4, -3\pi/4 \pm m\pi)$ where m is an integer.

The same conditions result for cancelling the crosstalk between signals sent at a center frequency of $\pi - 3\pi/N$ and the other end frequency $\pi - \pi/N$.

Although the preceding analysis generates only one type of solution, there are in fact two embedded solutions that arise by making the difference in the delay factors an odd or even multiple of $N/2$.

3.4.2 Crosstalk: Repeated Center Frequencies

Here, we examine the crosstalk function associated with two signals transmitted with the same center frequency. We return to the original expression for the crosstalk function as in Eq. (3.7) and let ω_l be equal to ω_k to get

$$T_{kl}(z^N) = \frac{1}{4} z^{-(n_k - p_l)} \sum_{i=0}^{N-1} W^{i(n_k - p_l)} \Big[e^{j(\alpha_k + \beta_l)} H^2(e^{-j\omega_k} z W^{-i}) + e^{-j(\alpha_k + \beta_l)} H^2(e^{j\omega_k} z W^{-i}) + 2\cos(\alpha_k - \beta_l) H(e^{j\omega_k} z W^{-i}) H(e^{-j\omega_k} z W^{-i}) \Big] . \quad (3.11)$$

In this specific case, the crosstalk function $T_{kl}(z^N)$ is comprised of three crosstalk terms. For $2\pi/N \le \omega_k \le \pi - 2\pi/N$, the third crosstalk term in the above equation is zero due to the bandlimitedness of $H(z)$. The crosstalk function is reduced to

$$T_{kl}(z^N) = \frac{1}{4} z^{-(n_k - p_l)} \sum_{i=0}^{N-1} W^{i(n_k - p_l)} \Big[e^{j(\alpha_k + \beta_l)} H^2(e^{-j\omega_k} z W^{-i}) + e^{-j(\alpha_k + \beta_l)} H^2(e^{j\omega_k} z W^{-i}) \Big] . \quad (3.12)$$

We have many degrees of freedom with which to force a zero crosstalk function. To maintain compatibility with the types of solutions formulated earlier, we restrict the differences in the delays to be multiples of $N/2$ and the sum of the phases to be multiples of $\pi/2$. Otherwise, we admit the situation of deriving conditions which when united with the specific solutions in Sections 3.3 and 3.4.1 become contradictory in that no combination of the parameters would satisfy the entire set. Given the delays and phases as above, the analysis procedure investigates the question of which center frequencies can be utilized for transmitting more than one signal. The details are laid out in Appendix C. Given the derivations in Appendix C, we have the following restrictions on the center frequencies.

1. If $n_k - p_l$ is a multiple of N and $\alpha_k + \beta_l$ is an odd multiple of $\pi/2$, the center frequency must be a multiple of π/N.
2. If $n_k - p_l$ is an odd multiple of $N/2$ and $\alpha_k + \beta_l$ is a multiple of π, the center frequency must be an odd multiple of π/N.

3. If $n_k - p_l$ is an odd multiple of $N/2$ and $\alpha_k + \beta_l$ is an odd multiple of $\pi/2$, the center frequency must be an even multiple of π/N.

The crosstalk cannot be made zero if $n_k - p_l$ is a multiple of N and $\alpha_k + \beta_l$ is a multiple of π.

It was initially established that the repeated center frequencies are multiples of π/N. Here, we have an additional result that fixes these frequencies. It has been shown that with appropriate limitations on the delays and phases, the repeated center frequencies must be multiples of π/N to ensure zero crosstalk.

The preceding analysis is specifically devoted to the center frequencies in the closed interval $[2\pi/N, \pi - 2\pi/N]$. The remaining case that must be considered is the end center frequency π/N in Set 2. Two signals can be transmitted at this frequency without crosstalk subject to both conditions given below.

1. The delays are chosen such that $n_k - p_l$ is an odd multiple of $N/2$.
2. The phases are chosen such that $(\alpha_k, \beta_l) = (\pi/4, -\pi/4 \text{ or } 3\pi/4)$, $(-\pi/4, \pi/4 \text{ or } -3\pi/4)$, $(3\pi/4, -3\pi/4 \text{ or } \pi/4)$ or $(-3\pi/4, 3\pi/4 \text{ or } -\pi/4)$.

The same conditions hold for the other end frequency of $\pi - \pi/N$ in Set 2.

3.4.3 Distinct Center Frequencies of Set 3

Now, we consider the distinct center frequencies of Set 3. Crosstalk due to spectral overlap occurs only between two signals transmitted at adjacent center frequencies. In Set 3, let two adjacent center frequencies be given by $\omega_k = (2r+1)\pi/2N$ and $\omega_l = (2r+3)\pi/2N$ for $r = 0, 1, \cdots, N-2$. By substituting these frequencies in Eq. (3.7), invoking the bandlimitedness assumption for $H(z)$ and performing algebraic manipulation just as in Section 3.4.1 gives a relationship similar to Eq. (3.10) as

$$\alpha_k + \beta_l = \pi \left[\frac{(r+1)(n_k - p_l)}{N} + \frac{1}{2} \right] . \tag{3.13}$$

Note that the same relationship holds between $\alpha_l + \beta_k$ and $n_l - p_k$.

Just like Eq. (3.10), Eq. (3.13) depicts a general relationship between two unknowns $\alpha_k + \beta_l$ and $n_k - p_l$. In contrast to the situation of having repeated center frequencies, only one type of solution to Eq. (3.13) is necessary since the center frequencies are distinct. This is provided without any restriction on N by setting $n_k - p_l$ to be a multiple of N. However, we can maintain the principle of making $n_k - p_l$ a rational multiple of N and impose the mild limitation of an even N to get a second type of solution (similar to the approach in Section 3.4.1). The two types of solutions lead to two different transmultiplexers.

Solution One

1. The delays are chosen such that $n_k - p_l$ and $n_l - p_k$ are multiples of N.
2. The phases are chosen such that $\alpha_k + \beta_l$ and $\alpha_l + \beta_k$ are odd multiples of $\pi/2$.

Solution Two

1. The delays are chosen such that $n_k - p_l$ and $n_l - p_k$ are odd multiples of $N/2$.
2. If r is odd, $\alpha_k + \beta_l$ and $\alpha_l + \beta_k$ are odd multiples of $\pi/2$. If r is even, $\alpha_k + \beta_l$ and $\alpha_l + \beta_k$ are multiples of π.

3.5 Synthesized Transmultiplexers

The specific solutions proposed in Sections 3.3, 3.4.1, 3.4.2 and 3.4.3 comprise a set of sufficient conditions for an N band crosstalk-free transmultiplexer with an identical input-output transfer function for every pair of corresponding terminals. Given these solutions, we establish values for the free parameters and synthesize five different types of transmultiplexers. The first three use repeated center frequencies (Set 1 or 2). The other two use the distinct frequencies of Set 3. In four of the five systems, it is necessary to implement delays which are odd multiples of $N/2$. For these cases, the parameter N is constrained to be even.

3.5.1 System T1

In the first system T1, we use center frequencies in Set 1. The combining and separation filters corresponding to the end frequency $\omega_0 = 0$ are associated with parameters $n_0 = p_0 = 0$ and $\alpha_0 = -\beta_0 = \pi/4$. The next center frequency, $\omega_1 = \omega_2 = 2\pi/N$ is used to transmit two signals. Crosstalk is eliminated between these two signals and with the signal sent at zero frequency by setting $n_1 = p_1 = N/2$, $\alpha_1 = -\beta_1 = \pi$, $n_2 = p_2 = 0$ and $\alpha_2 = -\beta_2 = -\pi/2$. Now, we proceed to the frequency $\omega_3 = \omega_4 = 4\pi/N$. To cancel crosstalk between signals sent at $2\pi/N$ and $4\pi/N$, we set $n_3 = p_3 = 0$, $\alpha_3 = -\beta_3 = 0$, $n_4 = p_4 = N/2$ and $\alpha_4 = -\beta_4 = 3\pi/2$. These parameters eliminate crosstalk between the two signals sent at $4\pi/N$ due to the compatibility in the rules formed for cancelling crosstalk due to spectral overlap between adjacent and repeated frequencies. We continue this procedure in a sequential fashion for each center frequency. This establishes the combining and separation filters of T1 as follows:

$$
\begin{array}{ll}
a_0(n) = h(n)\cos\frac{\pi}{4} & b_0(n) = h(n)\cos\frac{\pi}{4} \\
\left\{\begin{array}{l} a_1(n) = h(n-\frac{N}{2})\cos\frac{2\pi}{N}n \\ a_2(n) = h(n)\sin\frac{2\pi}{N}n \end{array}\right. & \begin{array}{l} b_1(n) = h(n+\frac{N}{2})\cos\frac{2\pi}{N}n \\ b_2(n) = -h(n)\sin\frac{2\pi}{N}n \end{array} \\
\left\{\begin{array}{l} a_3(n) = h(n)\cos\frac{4\pi}{N}n \\ a_4(n) = h(n-\frac{N}{2})\sin\frac{4\pi}{N}n \end{array}\right. & \begin{array}{l} b_3(n) = h(n)\cos\frac{4\pi}{N}n \\ b_4(n) = -h(n+\frac{N}{2})\sin\frac{4\pi}{N}n \end{array} \\
\vdots \quad \vdots & \vdots \quad \vdots
\end{array}
\tag{3.14}
$$

It is noted that for T1, the delay elements of $N/2$ alternate between the cosine and sine carriers and that the separation filters associated with the sine carriers have a minus sign associated with $h(n)$. It is also observed that a delay element of $N/2$ is associated with a center frequency of π only if $N = 2, 6, 10, \ldots$. The input-output transfer function for any pair of corresponding terminals is

$$
\begin{aligned}
T(z^N) &= \frac{1}{2}\sum_{i=0}^{N-1} H^2(zW^{-i}) \\
&= \frac{N}{2}\left[\cdots + v(-2N)z^{2N} + v(-N)z^N + v(0) + v(N)z^{-N} + v(2N)z^{-2N} + \cdots\right]
\end{aligned}
\tag{3.15}
$$

where $v(n)$ is the inverse z-transform of $H^2(z)$.

3.5.2 System T2

In the second system T2, we use center frequencies in Set 2. The combining and separation filters for the signals sent with the end center frequency π/N have parameters $n_0 = p_0 = 0$, $\alpha_0 = -\beta_0 = \pi/4$, $n_1 = p_1 = N/2$ and $\alpha_1 = -\beta_1 = \pi/4$. There is no crosstalk between the signals transmitted at π/N. For a frequency of $3\pi/N$, crosstalk due to spectral overlap with π/N is cancelled by setting $n_2 = p_2 = 0$, $\alpha_2 = -\beta_2 = -\pi/4$, $n_3 = p_3 = N/2$ and $\alpha_3 = -\beta_3 = 7\pi/4$. We observe that these parameters ensure no crosstalk between the signals sent at $3\pi/N$. For the next frequency $5\pi/N$, crosstalk due to spectral overlap with $3\pi/N$ is cancelled by invoking the type of solution derived in Section 3.4.1. Again, these parameters eliminate crosstalk arising from frequency repetition due to the compatability of the derived conditions. This process continues in a sequential fashion. This establishes the combining and separation filters of T2 as follows:

$$\begin{aligned}
a_0(n) &= h(n)\cos\left(\frac{\pi}{N}n + \frac{\pi}{4}\right) & b_0(n) &= h(n)\cos\left(\frac{\pi}{N}n - \frac{\pi}{4}\right) \\
a_1(n) &= h\left(n - \frac{N}{2}\right)\cos\left(\frac{\pi}{N}n - \frac{\pi}{4}\right) & b_1(n) &= h\left(n + \frac{N}{2}\right)\cos\left(\frac{\pi}{N}n + \frac{\pi}{4}\right) \\
a_2(n) &= h(n)\cos\left(\frac{3\pi}{N}n - \frac{\pi}{4}\right) & b_2(n) &= h(n)\cos\left(\frac{3\pi}{N}n + \frac{\pi}{4}\right) \\
a_3(n) &= h\left(n - \frac{N}{2}\right)\cos\left(\frac{3\pi}{N}n + \frac{\pi}{4}\right) & b_3(n) &= h\left(n + \frac{N}{2}\right)\cos\left(\frac{3\pi}{N}n - \frac{\pi}{4}\right) \\
\vdots \quad & \quad \vdots & \vdots \quad & \quad \vdots
\end{aligned} \tag{3.16}$$

The delay element of $N/2$ alternates between the cosine carriers having a resultant phase of $-\pi/4$ and $\pi/4$. When no delay element is present, the resultant phase of the cosine carriers alternates between $\pi/4$ and $-\pi/4$. The input-output transfer function for any pair of corresponding terminals is

$$\begin{aligned}
T(z^N) &= \frac{1}{2}\sum_{i=0}^{N-1} H^2(zW^{-i+\frac{1}{2}}) \\
&= \frac{N}{2}\left[\cdots + v(-2N)z^{2N} - v(-N)z^N + v(0) - v(N)z^{-N} + v(2N)z^{-2N} - \cdots\right]
\end{aligned} \tag{3.17}$$

where $v(n)$ is the inverse z-transform of $H^2(z)$.

3.5.3 System T3

A third transmultiplexer is synthesized by relaxing the assumption of using only a single lowpass prototype. The system uses two lowpass prototypes $h(n)$ and $g(n)$ which are each bandlimited to no less than π/N and no more than $2\pi/N$. Each of the combining and separation filters is a modulated and delayed version of one of the lowpass prototypes just as in Eqs. (3.1) and (3.2).

Suppose T1 is modified to include two prototypes by alternating the positions of $h(n)$ and $g(n)$ between the combining and separation filters for each center frequency. This leads to a new transmultiplexer T3 described as follows.

$$\begin{aligned}
a_0(n) &= h(n)\cos\frac{\pi}{4} & b_0(n) &= g(n)\cos\frac{\pi}{4} \\
a_1(n) &= g(n-\frac{N}{2})\cos\frac{2\pi}{N}n & b_1(n) &= h(n+\frac{N}{2})\cos\frac{2\pi}{N}n \\
a_2(n) &= g(n)\sin\frac{2\pi}{N}n & b_2(n) &= -h(n)\sin\frac{2\pi}{N}n \\
a_3(n) &= h(n)\cos\frac{4\pi}{N}n & b_3(n) &= g(n)\cos\frac{4\pi}{N}n \\
a_4(n) &= h(n-\frac{N}{2})\sin\frac{4\pi}{N}n & b_4(n) &= -g(n+\frac{N}{2})\sin\frac{4\pi}{N}n \\
\vdots \quad & \quad \vdots & \vdots \quad & \quad \vdots
\end{aligned} \tag{3.18}$$

The crosstalk between two signals sent at adjacent frequencies is eliminated as in T1. Moreover, it can be shown that the crosstalk between two signals transmitted at the same center frequency ω_k where $2\pi/N \leq \omega_k \leq \pi - 2\pi/N$ is eliminated by the same approaches as derived in Section 3.4.2 even when two prototypes are used. Therefore, system T3 is crosstalk-free. The input-output transfer function for any pair of corresponding terminals is

$$\begin{aligned}T(z^N) &= \frac{1}{2}\sum_{i=0}^{N-1} H(zW^{-i})G(zW^{-i}) \\ &= \frac{N}{2}\Big[\cdots + w(-2N)z^{2N} + w(-N)z^N + w(0) \\ &\qquad + w(N)z^{-N} + w(2N)z^{-2N} + \cdots\Big]\end{aligned} \tag{3.19}$$

where $w(n)$ is the inverse z-transform of $H(z)G(z)$.

Consider modifying T2 to include two prototypes. Again, we alternate the positions of $h(n)$ and $g(n)$ between the combining and separation filters for each center frequency. In this case, the crosstalk between two signals sent at an end center frequency (π/N or $(N-1)\pi/N$) is not cancelled with two prototypes. Therefore, T2 cannot be modified to include two prototypes.

Note that there are other ways of modifying the transmultiplexers to allow for two prototypes. However, any other arrangement leads to a crosstalk function $T_{kl}(z^N)$ for two signals sent at adjacent frequencies to be expressed in terms of $H(z)$ and $G(z)$. Then, the crosstalk terms in $T_{kl}(z^N)$ that involve spectral overlap cannot be cancelled. To conclude, we observe that only T1 can be modified to allow for two prototypes. Moreover, the modification must be performed in the unique way described above.

3.5.4 System T4

The center frequencies of Set 3 are used to synthesize system T4. A lowpass prototype with a maximum bandwidth of π/N is used. One of the specific solutions formulated in Section 3.4.3 is invoked to establish the parameters. All the delay factors n_k and p_k equal zero. The phases are set such that $(\alpha_k, \beta_k) = (-\pi/4, \pi/4)$ for $k = 0, 2, \cdots$ and $(\alpha_k, \beta_k) = (\pi/4, -\pi/4)$ for $k = 1, 3, \cdots$. The filter bank description of T4 is as follows:

$$\begin{aligned}
a_0(n) &= h(n) \cos\left(\frac{\pi}{2N}n - \frac{\pi}{4}\right) & b_0(n) &= h(n) \cos\left(\frac{\pi}{2N}n + \frac{\pi}{4}\right) \\
a_1(n) &= h(n) \cos\left(\frac{3\pi}{2N}n + \frac{\pi}{4}\right) & b_1(n) &= h(n) \cos\left(\frac{3\pi}{2N}n - \frac{\pi}{4}\right) \\
a_2(n) &= h(n) \cos\left(\frac{5\pi}{2N}n - \frac{\pi}{4}\right) & b_2(n) &= h(n) \cos\left(\frac{5\pi}{2N}n + \frac{\pi}{4}\right) \\
&\vdots \quad \vdots & &\vdots \quad \vdots
\end{aligned} \tag{3.20}$$

The input-output transfer function for any pair of corresponding terminals is

$$\begin{aligned}
T(z^N) &= \frac{1}{4} \sum_{i=0}^{N-1} [H^2(zW^{-i+\frac{1}{4}}) + H^2(zW^{-i-\frac{1}{4}})] \\
&= \frac{N}{2} \Big[\cdots + v(-4N)z^{4N} - v(-2N)z^{2N} + v(0) \\
&\qquad - v(2N)z^{-2N} + v(4N)z^{-4N} - \cdots \Big] .
\end{aligned} \tag{3.21}$$

Note that the transfer function is in fact a function of z^{2N}. In fact, system T4 is the same as the transmultiplexer formed as the complement of the subband system in [10] except for the phase factors. The complement of the system in [10] has phase factors which satisfy the same solution in Section 3.4.3 that was invoked in forming T4 and which are either the same as those in T4 or differ from those in T4 by a multiple of π. Therefore, the synthesis procedure includes an existing modulated filter bank based on one prototype and equally spaced distinct center frequencies.

Just as T1 was modified to get T3, we attempt to modify T4 by alternating the positions of two prototypes (both bandlimited to no more than π/N) between the combining and separation filters for each center frequency. With this arrangement, the crosstalk between two signals sent at adjacent center frequencies is cancelled as in T4. However, it can be shown that the input-output transfer function will not be the same for all pairs of terminals.

3.5.5 System T5

In system T5, we again use the center frequencies of Set 3 and bandlimit the prototype to no more than π/N. Another type of solution formulated in Section 3.4.3 is used to configure T5. For the end center frequency $\pi/2N$, the parameters are $n_0 = p_0 = 0$ and $\alpha_0 = -\beta_0 = -\pi/4$. When establishing the combining and separation filters for the frequency $3\pi/2N$, crosstalk due to spectral overlap with $\pi/2N$ is cancelled by setting $n_1 = p_1 = N/2$ and $\alpha_1 = -\beta_1 = 3\pi/4$. This process continues in a sequential fashion such that the filter bank of T5 is described by:

$$
\begin{aligned}
a_0(n) &= h(n)\cos(\frac{\pi}{2N}n - \frac{\pi}{4}) & b_0(n) &= h(n)\cos(\frac{\pi}{2N}n + \frac{\pi}{4}) \\
a_1(n) &= h(n - \frac{N}{2})\cos\frac{3\pi}{2N}n & b_1(n) &= h(n + \frac{N}{2})\cos\frac{3\pi}{2N}n \\
a_2(n) &= h(n)\cos(\frac{5\pi}{2N}n + \frac{\pi}{4}) & b_2(n) &= h(n)\cos(\frac{5\pi}{2N}n - \frac{\pi}{4}) \\
a_3(n) &= h(n - \frac{N}{2})\sin\frac{7\pi}{2N}n & b_3(n) &= -h(n + \frac{N}{2})\sin\frac{7\pi}{2N}n \\
&\vdots \quad \vdots & &\vdots \quad \vdots
\end{aligned}
\tag{3.22}
$$

System T5 is a new alternative to T4. A delay element of $N/2$ is present for every other center frequency. In addition, the modulating function alternates between a cosine and a sine for the cases in which a delay element is present. When no delay element is present, the resultant phase of the cosine carrier alternates between $-\pi/4$ and $\pi/4$. The input-output transfer function for T5 is the same as that for T4 and is given by Eq. (3.21).

As for T4, we attempt to modify T5 to accommodate two prototypes. In general, the transfer function is not the same between every pair of corresponding terminals. However, an exception occurs for the case $N = 2$ (see Section 3.7).

3.5.6 Modification of the Parameters

We address the question of whether the parameters chosen from the specific solutions proposed in Sections 3.3, 3.4.1, 3.4.2 and 3.4.3 to configure the transmultiplexers are unique. For the cases when the delay element is 0, a delay factor of the form aN/b where a and b are relatively prime could be used. Then, the delay elements of $N/2$ would be replaced by $aN/b + N/2 = (2a + b)N/2b$. In the general case, when $2a + b$ and $2b$ are relatively prime, N is restricted to be an integral multiple of $2b$ in order to realize integral delay factors. Otherwise, N must be an integral multiple of b. Our choice of delay factors imposes the most mild restriction on N in that N must be even.

Any set of phase factors (α_k, β_k) can be replaced by $(\alpha_k + m\pi, \beta_k - m\pi)$ where m is an integer. Also, note that the transmultiplexers T1 through T5 were developed by starting with phase factors $\pi/4$ and $-\pi/4$ for the lowest end frequency. However, any odd multiple of $\pi/4$ could be used as the starting point. In this case, the filters in T1 and T3 are either the same as or negatives of those presented above. In systems T2, T4 and T5, the phase factors used in all the filters would still remain to be odd multiples of $\pi/4$. The cosine and sine carriers at the odd-indexed terminals of T5 could become sine and cosine carriers respectively. To conclude, we note that the chosen parameters are not unique. However, changes in the delay factors will restrict the number of bands and adjustments of the phase factors render only a trivial modification of the system. Finally, note that the swapping property (discussed in Chapter 2) applies to the synthesized transmultiplexers but offers no significant change in the delay and phase factors.

3.5.7 Elimination of Intersymbol Interference

The five preceding transmultiplexers have been synthesized to eliminate crosstalk. The input-output transfer function $T(z^N)$ still admits intersymbol interference. Intersymbol interference is eliminated in T1 and T2 if $H^2(z)$ satisfies the Nyquist criterion in which every Nth sample of its impulse response (except for a reference sample) is equal to zero. In T3, the product $H(z)G(z)$ must satisfy the Nyquist criterion. Intersymbol interference is eliminated in T4 and T5 if $H^2(z)$ satisfies the Nyquist criterion in which every $2N$th sample of its impulse response (except for a reference sample) is equal to zero. Perfect reconstruction is achieved if the prototypes are bandlimited (up to hundred percent above the minimum bandwidth) and the Nyquist criterion is satisfied. The problem of designing the prototypes to satisfy the Nyquist criterion is addressed in Chapters 4 and 5.

3.6 Multicarrier QAM and VSB Systems

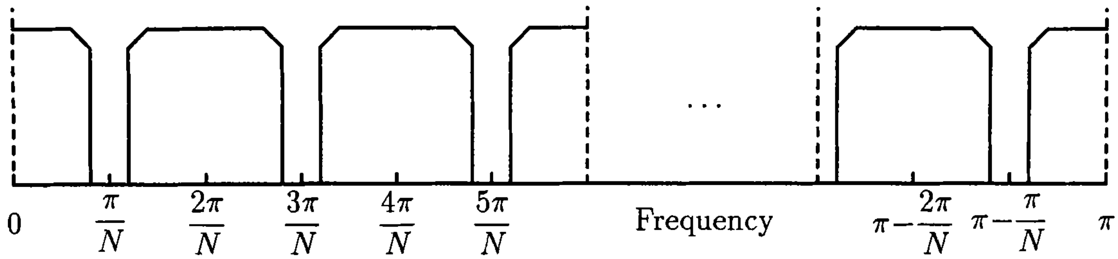
An interpretation of what exactly each system implements is made clear by examining the input signal spectrum and the filter responses as shown in Fig. 3.1. As shown in Fig. 3.1(a), modulation is implicitly accomplished by the sampling rate expander in that copies of the input signal spectrum appear at intervals of $2\pi/N$. The three systems T1, T2 and T3 accomplish multicarrier Quadrature Amplitude Modulation (QAM) in the form of a digital multirate filter structure as in Fig. 2.1. For each unique center frequency (except 0 and π), two signals are sent in quadrature. Systems T1 and T3 explicitly accomplish QAM in that a particular combining filter extracts one of the replicated copies of the input spectrum around carrier frequencies at multiples of $2\pi/N$ (see Fig. 3.1). The same is not true of T2 in that the combining filters, whose center frequencies are odd multiples of π/N, extract a portion of two adjacent copies of the input spectrum. System T2 can be converted to a true QAM scheme as follows. Suppose each of the input signals is multiplied by $(-1)^n$ prior to sampling rate expansion. Then, the input spectrum shifts in such a way that after sampling rate expansion, the replicated copies are centered at implicit carriers equal to odd multiples of π/N (shown in Fig. 3.1(b)). Now, each of the combining filters will extract a replicated copy centered at an odd multiple of π/N. The original signals can be recovered by multiplying each of the outputs by $(-1)^n$.

Multicarrier Quadrature Amplitude Modulation systems have been realized in continuous time [21] and in discrete time [22]. Also, a data modem based on the QAM technique is described in [23]. The system in [22] uses one lowpass prototype and a set of equally spaced frequencies for transmission. Also, it is oversampled as opposed to the critically sampled systems that we consider. In an oversampled system, the sampling rate expansion/compression factor is greater than the number of frequency bands. This gives additional freedom in choosing the repeated center frequencies but does not generally result in the utilization of the entire range 0 to π. In every band, the lowpass prototype performs an interpolation function by extracting the copy of the input spectrum around the 0 frequency. Each of the filtered outputs is then explicitly modulated by multiplication with a sinusoid at the corresponding carrier frequency. Two signals are sent in quadrature at each carrier frequency through modulation by a cosine and sine carrier. Our system T1 is related to the system proposed in [22]. The system in [22] can be converted to our system T1 with the restriction that the carrier frequencies are multiples of $2\pi/N$.

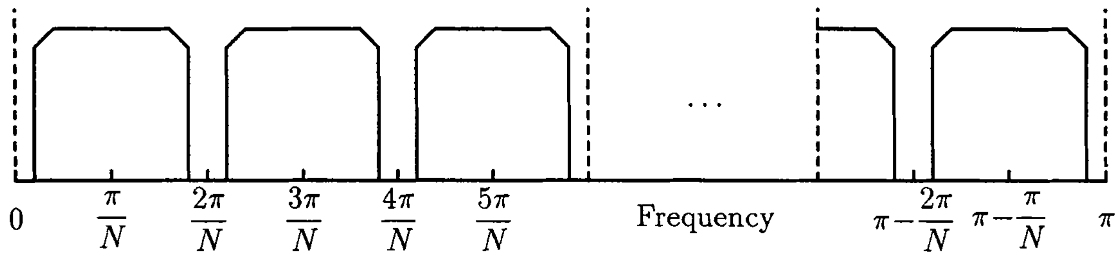
In contrast, system T4 and system T5 do not implement QAM. Systems T4 and T5 can be thought of as being multicarrier Vestigial Sideband (VSB) schemes. Given an implicit set of carriers at multiples of $2\pi/N$, there are both lower and upper sidebands at multiples of $2\pi/N$. A combining filter extracts either an upper or lower sideband of a particular copy of the input spectrum and a vestige of a suppressed sideband for transmission. Multiplication of the input signal by $(-1)^n$ prior to sampling rate expansion results in an implicit set of carriers at odd multiples of π/N. Again, one upper or lower sideband and a vestige of a suppressed sideband is extracted for transmission. In contrast to conventional frequency division multiplexing (FDM) schemes which avoid spectral overlap by using guard bands, the VSB systems allow overlap between the transmitted sidebands of different input signals.

Another multirate system described in [9] is not a regular structure in that the center frequencies are not equally spaced and two prototypes of different bandwidths are used to derive the filter banks. Although the system in [9] is a subband system, it can be converted into a transmultiplexer. When viewed as a transmultiplexer, the system in [9] implements VSB for all carriers other than 0 and π.

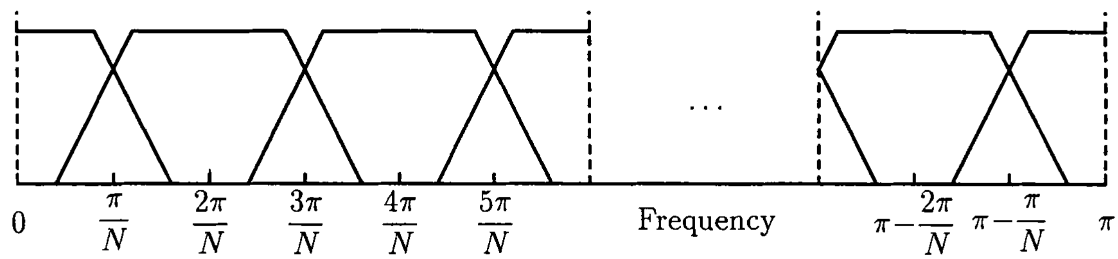
A synthesis procedure that establishes a set of analog transmitter filters for the



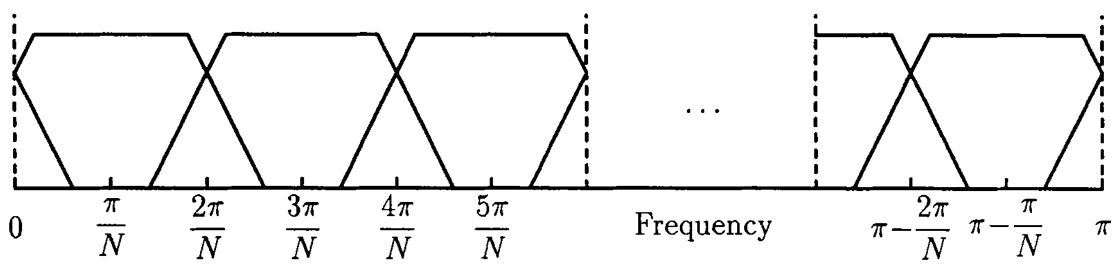
(a) Input signal spectrum after sampling rate expansion



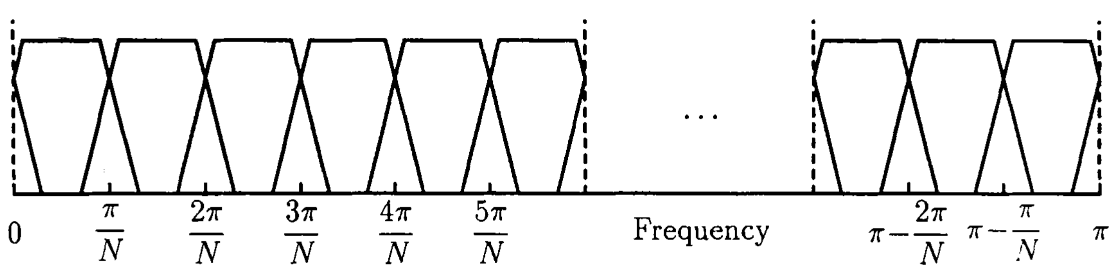
(b) Spectrum of input signal multiplied by $(-1)^n$ after sampling rate expansion



(c) Filter responses for systems T1 and T3



(d) Filter responses for system T2



(e) Filter responses for systems T4 and T5

Fig. 3.1 Input signal spectrum and responses of the filters used in systems T1 to T5 (shown for N even) simultaneous transmission of data is developed in [24]. The approach in [24] consists of simultaneously deriving conditions on the amplitude and phase characteristics of the filters such that crosstalk and intersymbol interference are cancelled. This leads to a class of multicarrier analog transmission systems. In this thesis, an equivalent class of digital systems are configured. In contrast to the method in [24], our synthesis procedure decouples the problems of crosstalk and intersymbol interference. The parameters of the cosine modulating function allow for crosstalk due to spectral overlap to be cancelled. Intersymbol interference is eliminated by designing the lowpass prototype $h(n)$ (discussed in Chapter 4). Transmultiplexer T4 is a digital counterpart to the system configured in [24].

3.7 The Two Band Case

This section examines two band systems as a separate case. Although two band versions of transmultiplexers T1 to T5 exist, we anticipate that a synthesis procedure devoted only to the $N = 2$ case will lead to more flexible conditions than the N band case and consequently, lead to many transmultiplexers. As before, the combining filters $A_k(z)$ have parameters ω_k, n_k and α_k for $k = 0$ and 1. The separation filters $B_k(z)$ have parameters ω_k, p_k and β_k for $k = 0$ and 1. We do not impose any bandwidth restriction on the lowpass prototypes in formulating a synthesis procedure for crosstalk-free transmultiplexers with two identical input-output transfer functions.

For systems based on one prototype filter and with two distinct center frequencies, the following conditions must hold.

1. The two center frequencies must satisfy the relation $\omega_0 + \omega_1 = \pi$.

2. The delays are chosen such that:

   (i) The relationship $n_0 - p_0 = n_1 - p_1$ must be satisfied. Moreover, both $n_0 - p_0$ and $n_1 - p_1$ are even.

   (ii) Both $n_0 - p_1$ and $n_1 - p_0$ are odd.

3. The phases are chosen such that:

   (i) If $\omega_0 \neq 0$ and $\omega_1 \neq \pi$, then $\alpha_0 + \beta_0 = -(\alpha_1 + \beta_1)$. If $\omega_0 = 0$ and $\omega_1 = \pi$, then $\alpha_0 + \beta_0 = \pm(\alpha_1 + \beta_1)$.

   (ii) The relationship $\alpha_0 - \beta_0 = \pm(\alpha_1 - \beta_1)$ must hold.

   (iii) If $\omega_0 \neq 0$ and $\omega_1 \neq \pi$, both $\alpha_0 + \beta_1$ and $\alpha_1 + \beta_0$ are multiples of π.

For the case in which both center frequencies are the same, we have the same restrictions on the delays as given above. The center frequency is $\pi/2$. Appendix D justifies this choice. The restrictions on the phases are as above except that 3(i) becomes $\alpha_0 + \beta_0 = \pm(\alpha_1 + \beta_1)$.

Now, consider the case when two prototypes $H(z)$ and $G(z)$ are used. The filters $A_0(z)$ and $B_1(z)$ are frequency shifted versions of $H(z)$. Similarly, $A_1(z)$ and $B_0(z)$ are frequency shifted versions of $G(z)$. The conditions for the cancellation of crosstalk remain the same as above. The input-output transfer function is examined to establish any further requirements. For distinct center frequencies except 0 and π, the rules are the same as for the single prototype case except that 3(ii) changes to $\alpha_0 - \beta_0 = \alpha_1 - \beta_1$. If $\omega_0 = 0$ and $\omega_1 = \pi$, the rules are the same as the single prototype case. For the case in which the center frequencies are the same, the rules are again the same as the single prototype case.

As anticipated, the above rules permit the synthesis of many two band transmultiplexers. There is no bandwidth restriction on the prototypes for the two band case. This allows for more freedom in choosing the center frequencies for the 2 band case as compared to the N band case and yet ensures complete bandwidth utilization. Table 3.1 shows some two band systems that are synthesized from the formulated rules.

System	Center Frequencies	Combining Filters	Separation Filters
A	0	$a_0(n) = h(n)$	$b_0(n) = g(n)$
	π	$a_1(n) = (-1)^n g(n-1)$	$b_1(n) = (-1)^n h(n+1)$
B	$\pi/2$	$a_0(n) = h(n)\cos(\frac{\pi}{2}n + \frac{\pi}{4})$	$b_0(n) = g(n)\cos(\frac{\pi}{2}n - \frac{\pi}{4})$
	$\pi/2$	$a_1(n) = g(n-1)\cos(\frac{\pi}{2}n - \frac{\pi}{4})$	$b_1(n) = h(n+1)\cos(\frac{\pi}{2}n + \frac{\pi}{4})$
C	$\pi/4$	$a_0(n) = h(n)\cos(\frac{\pi}{4}n - \frac{\pi}{4})$	$b_0(n) = g(n)\cos(\frac{\pi}{4}n + \frac{\pi}{4})$
	$3\pi/4$	$a_1(n) = g(n-1)\cos(\frac{3\pi}{4}n)$	$b_1(n) = h(n+1)\cos(\frac{3\pi}{4}n)$
D	$\pi/3$	$a_0(n) = h(n)\cos(\frac{\pi}{3}n + \frac{\pi}{3})$	$b_0(n) = g(n)\cos(\frac{\pi}{3}n - \frac{\pi}{3})$
	$2\pi/3$	$a_1(n) = g(n-1)\cos(\frac{2\pi}{3}n - \frac{\pi}{3})$	$b_1(n) = h(n+1)\cos(\frac{2\pi}{3}n + \frac{\pi}{3})$

Table 3.1 Synthesized Two Band Systems

The systems depicted in Table 3.1 involve two prototypes. One prototype versions occur as a special case. System A is a two band version of T3 (the two band version of T1 is the special case). When $G(z) = H(z)$, System B is a two band version of T2. Although many two band systems can be developed, they cannot necessarily be extended to the N band case for our objectives. An N band version of System B cannot be configured since the crosstalk function for two signals sent at adjacent center frequencies will involve two prototypes and cannot be made equal to zero. If $G(z) = H(z)$, an N band version of System C results if the bandwidth of the prototype is reduced to π/N (system T5). However, an N band system with two prototypes cannot be formed even with the reduced bandwidth since the input-output transfer function is not the same for every pair of terminals. System D is synthesized by taking advantage of the flexibility in choosing the center frequencies specifically for the two band case. The general synthesis procedure in this chapter does not lead to an N band version of System D even if $G(z) = H(z)$.

3.8 Subband Complements

Transmultiplexers T1 through T5 are crosstalk-free. Moreover, each transmultiplexer has an identical input-output transfer function for every pair of corresponding terminals. Therefore, systems T1 to T5 can be converted into alias-free subband systems S1, S2, S3, S4 and S5 respectively (this complementary nature of the two multirate systems was discussed in Chapter 2). The new subband systems S1, S2 and S3 have repeated center frequencies. The subband systems S4 and S5 have the same distinct center frequencies. System S4 resembles the one in [10] while S5 is an alternative employing delay factors. Note that the other two band transmultiplexers that are synthesized in Section 3.7 can also be converted into subband systems.

The input-output transfer function for the transmultiplexers is given as $T(z^N)$ in Eqs. (3.15), (3.17), (3.19) and (3.21). For the transmultiplexers, the input-output relationship is $\hat{X}_k(z) = \frac{1}{N}T(z)X_k(z)$ for $k = 0$ to $N-1$. For the complementary subband systems, $\hat{X}(z) = \frac{1}{N}T(z^N)X(z)$. Note that this input-output relationship is dependent on bandlimited prototypes with a restricted stopband edge ω_s as for the transmultiplexers. Moreover, perfect reconstruction is achieved in the complementary subband systems if the Nyquist criterion is satisfied (as for the transmultiplexers).

We have configured transmultiplexers and subband systems that achieve perfect reconstruction given that the prototypes are bandlimited (up to hundred percent above the minimum bandwidth, as discussed earlier) and the Nyquist criterion is satisfied. When $G(z) = H(z^{-1})$ in systems T3 and S3, the two conditions of bandlimitedness and the Nyquist characteristic lead to perfect reconstruction with $B_k(z) = A_k(z^{-1})$. Therefore, both systems are lossless [8] under the same two conditions. For the special case of $N = 2$, S1 reduces to the classical QMF arrangement described by Eq. (2.10). Note that system S3 becomes the Smith-Barnwell structure [5] for the case $N = 2$ if $G(z) = H(z^{-1})$. For an arbitrary $H(z)$ and $G(z)$, system S3 degenerates into a general two band two prototype system proposed in [6][7] (also discussed in Chapter 2). In effect, we have developed subband systems which are N band generalizations of the QMF bank, the Smith-Barnwell structure and the general two band system proposed in [6][7] employing two prototypes.

Chapter 4 Minimax Filter Design

Given the synthesized transmultiplexers T1 to T5 and the complementary subband systems S1 to S5, we proceed to design the practical lowpass prototypes that describe the filter banks. In addition to the frequency response requirement, the time domain constraints on the impulse response of the prototypes that are needed to satisfy the Nyquist criterion are taken into account. The design of the prototypes is based on a minimax criterion.

4.1 Design Problem

The design problem of simultaneously satisfying the time and frequency domain constraints to yield Nyquist filters is introduced. Then, the general characteristics of Nyquist filters are described.

4.1.1 Time and Frequency Domain Requirements

For the QAM transmultiplexers T1 and T2 and their subband complements, the lowpass prototype $H(z)$ must be bandlimited to no less than π/N and no more than

$2\pi/N$. In addition, $H^2(z)$ must be a Nyquist filter with an impulse response having exact zero crossings every Nth sample (except for a reference sample). Similar requirements exist for T4, T5, S4 and S5 in that the lowpass $H(z)$ must be bandlimited to no less than $\pi/2N$ and no more than π/N. Also, $H^2(z)$ must be a Nyquist filter with an impulse response having exact zero crossings every $2N$th sample (except for a reference sample).

Systems T3 and S3 involve two lowpass prototypes $H(z)$ and $G(z)$. Both must be bandlimited to no less than π/N and no more than $2\pi/N$. Also, $H(z)G(z)$ must be a Nyquist filter with an impulse response having exact zero crossings every Nth sample (except for a reference sample). In carrying out the filter design, we set $G(z) = H(z^{-1})$. Then, both the prototypes have identical magnitude responses but different phase responses. A Nyquist filter $H(z)H(z^{-1})$ must be designed and split into a minimum phase component $H(z)$ and a maximum phase component $H(z^{-1})$.

4.1.2 Nyquist Filters

Since the design problem mandates Nyquist filters, some of their basic characteristics are introduced together with relevant terminology used in the remainder of the thesis. A Nyquist filter $F(z)$ has the following impulse response characteristic:

$$f(iK) = \begin{cases} \dfrac{1}{K} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases} . \tag{4.1}$$

The parameter K is the zero crossing interval in the time response $f(n)$ †. The

† Actually, $f(0)$ can be any constant. However, setting $f(0) = 1/K$ makes the right hand side of Eq. (4.2) equal to 1.

reference coefficient is $f(0)$[‡]. In the frequency domain, this corresponds to

$$\sum_{i=0}^{K-1} F(e^{j(\omega-2\pi i/K)}) = 1 \ . \tag{4.2}$$

The minimum bandwidth solution is an ideal lowpass filter bandlimited to π/K. We allow an excess bandwidth of $\beta\pi/K$ to bring the overall bandwidth to $(1+\beta)\pi/K$. The parameter β is the roll-off factor of $|F(e^{j\omega})|$. In the QAM systems (T1, T2 and T3) and their subband complements, the zero crossing interval K is equal to the number of bands N. The situation differs for the VSB systems (T4 and T5) and their subband complements in that $K = 2N$. To ensure that the lowpass prototype is bandlimited as discussed in Section 4.1.1, the parameter $\beta \leq 1$. In this case, only adjacent replicas of the spectrum of $F(e^{j\omega})$ (located at center frequencies that are multiples of $2\pi/K$) overlap. Also, the upper edge of the passband is $\omega_p = (1 - \beta)\pi/K$ and the lower edge of the stopband is $\omega_s = (1+\beta)\pi/K$. The ideal frequency characteristic is

$$|F(e^{j\omega})| = \begin{cases} 1 & \text{for} \quad 0 \leq |\omega| \leq \omega_p \quad \text{Passband} \\ 0 & \text{for} \quad \omega_s \leq |\omega| \leq \pi \quad \text{Stopband} \ . \end{cases} \tag{4.3}$$

The response of an ideal filter makes a symmetrical transition from the passband to the stopband passing through the value 0.5 at $\omega = \pi/K$.

We consider design approaches for a practical linear phase FIR Nyquist filter $F(z)$ that approximates the ideal magnitude characteristic. The passband edge ω_p and the stopband edge ω_s are as defined above. The general zero constellation of $F(z)$ involves real axis zeros which occur in pairs at $z = z_0$ and z_0^{-1}. Unit circle

‡ Note that the reference coefficient need not occur at the zeroth sample. We specify it at the zeroth sample for purposes of exposition.

zeros occur in complex conjugate pairs. The general complex zeros of $F(z)$ occur in groups of four at $z = z_0, z_0^\star, z_0^{-1}$ and $(z_0^{-1})^\star$. When $F(z) = H^2(z)$, all its zeros must occur as double order zeros and it must have an odd number of taps. For the case $F(z) = H(z)H(z^{-1})$, we refer to $F(z)$ as a factorable Nyquist filter. An FIR filter $F(z)$ is necessary to ensure stability of both its factors. Moreover, $F(z)$ is inherently zero-phase and has an odd number of taps. For $F(z)$ to be factorable into minimum and maximum phase parts $H(z)$ and $H(z^{-1})$ respectively, the additional constraint is that all of its zeros on the unit circle must occur as double order zeros.

Finally, note that although we deal with an $F(z)$ which may yield noncausal lowpass FIR prototypes, causality can be ensured in an actual implementation of the transmultiplexers and subband systems by applying appropriate delay factors (discussed in Chapter 2).

4.2 One Prototype Systems

For the transmultiplexers and subband complements which are based on one prototype, $F(z) = H^2(z)$. For a linear phase $F(z)$, $H(z)$ is a linear phase FIR filter. Consider systems T1, T2, S1 and S2. For these systems, N must be even. If $H(z)$ has an odd number of taps, an appropriate choice of filter delay results in the center or reference coefficient of $H^2(z)$ emerging at a time index which is a multiple of N. If $H(z)$ has an even number of taps, there is no choice of delay that allows the reference coefficient of $H^2(z)$ to emerge at a time index which is a multiple of N. For an $H(z)$ with an even number of taps, the reference coefficient of $H^2(z)$ never shows up in the expression for the input-output transfer function $T(z^N)$. For systems T4, T5, S4 and S5, it can also be shown that a linear phase $H(z)$ must have an odd number of taps. Therefore, a linear phase $H(z)$ is constrained to have an odd number of taps. For the remainder of the thesis, we design $H(z)$ such that the reference coefficient of $H^2(z)$ emerges at the zeroth sample.

The design problem mandates a lowpass $H(z)$ such that $H^2(z)$ is a Nyquist filter with exact zero crossings in its impulse response. It is now shown that both these time and frequency domain requirements cannot be met in general. The approach is to determine the time domain requirements on $H(z)$ for $F(z) = H^2(z)$ to exhibit a Nyquist characteristic.

First, we deal with the case when the zero crossing interval $K = 2$. Consider a zero-phase $h(n)$ which has $2L+1$ taps from $n = -L$ to L. Then, $f(n)$ has $4L+1$ taps from $n = -2L$ to $2L$. Also, $f(2i) = 0$ except for a reference coefficient $f(0) = 1/2$. Since $f(2L) = f(-2L) = 0$, it implies that $h(L) = h(-L) = 0$. The number of taps of $h(n)$ and $f(n)$ are reduced. Now, $h(n)$ has $2L - 1$ taps from $n = -(L-1)$ to $L - 1$ and $f(n)$ has $4L - 3$ taps from $n = -(2L-2)$ to $2L - 2$. Again, the end coefficients $f(2L-2) = f(-2L+2) = 0$ implying that $h(L-1) = h(-L+1) = 0$ thereby reducing the number of coefficients of $H(z)$ by two. This process continues and results in the unique solution $H(z) = 1/\sqrt{2}$. This trivial result is the only filter $H(z)$ that guarantees exact zero crossings in the response of $H^2(z)$ for $K = 2$.

Consider the case when $K > 2$. If the filters are short $(2L < K)$, then $H(z)$ will have more than one tap and will be free of any time domain constraints. However, the number of taps of $H(z)$ is not sufficient for an acceptable lowpass characteristic.

For longer filters ($2L > K$), many nonlinear constraints on $h(n)$ are imposed which may compromise the desired lowpass nature.

Given the difficulty of simultaneously meeting the time and frequency domain requirements, our strategy is to get a lowpass filter and only approximately satisfy the time domain constraints. Although the zero crossings in $f(n)$ are not exactly met, the response is kept small at the time indices at which the zero crossings should occur. There are closed form expressions for the frequency response of a Nyquist filter given $\beta \leq 1$, one of which is a raised cosine spectrum. A lowpass $H(z)$ is designed by the McClellan-Parks algorithm [25] to approximate the square root of a raised cosine spectrum with a given roll-off factor. We get a linear phase filter whose frequency response is a minimax approximation of the desired response. Both equiripple and nonequiripple lowpass prototypes are designed with appropriate weighting functions.

4.3 Two Prototype System

In contrast to the one prototype systems, the practical Nyquist filter $F(z) = H(z)H(z^{-1})$ for T3 and S3 can be designed such that $H(z)$ is a good lowpass filter and $F(z)$ has exact zero crossings in its impulse response. Since $F(z)$ is a zero-phase function, the reference coefficient is $f(0)$. Although $F(z)$ must have an odd number of taps, there are no constraints on the number of coefficients of $H(z)$ and $H(z^{-1})$. We develop two new approaches to design $F(z)$ known as factorable minimax methods. The two approaches use the McClellan-Parks algorithm [25] as a first step to control the stopband response. The subsequent step incorporates the time domain constraints by forcing zero crossings in the impulse response. This leads to a spectrum that exactly satisfies Eq. (4.2). With a response satisfying Eq. (4.2), an approximately zero stopband characteristic assures an approximately constant passband characteristic (assuming $\beta < 1$). A few iterations of the above steps produces a factorable Nyquist filter with a Chebyshev stopband response. The Nyquist filter designed by our approaches is fundamentally equiripple. A nonequiripple filter can be obtained by applying additional frequency weighting. Finally, the lowpass prototypes $H(z)$ and $H(z^{-1})$ are obtained from the Nyquist filter.

In the past, FIR Nyquist filters have been designed using linear programming techniques [26][27], by the eigenfilter approach [28][29], and by the use of the McClellan-Parks algorithm as an intermediate step [30][31][32][33]. The methods in [27][29][31] allow for the splitting of the filter into its minimum and maximum phase parts. Salazar and Lawrence [27] set up the design as a linear programming problem incorporating the time domain constraints. In addition, the frequency response of the filter is forced to be nonnegative in order that the minimum and maximum phase factorization be possible. Mintzer [31] deals exclusively with the case when the zero crossings occur for every second sample. In that paper, the frequency response of an unconstrained filter is offset to ensure that it becomes nonnegative. In [29], the eigenfilter concept is applied to obtain a Nyquist filter that is factorable into minimum and maximum phase parts.

Nyquist filters with Chebyshev stopband behaviour have been designed in [33] using a multistage structure. The focus in [33] is on a computationally efficient multistage implementation. However, the resulting filters are not necessarily factorable.

One can make these filters factorable by adding a positive constant to the frequency response (as in [31], see also [34]) to make it nonnegative. However, this fixup excessively reduces the stopband attenuation for nonequiripple filters.

In the factorable minimax methods, we directly achieve a nonnegative frequency response with controlled stopband characteristics. Furthermore, the polynomial factorization problem for the determination of the minimum phase part is considerably eased. The complexity of polynomial factorization is directly related to the order and hence, to the length of the designed Nyquist filter. We reduce this complexity by determining a partial factorization of the transfer function of the Nyquist filter as a byproduct of the design procedure. The remaining factorization involves a polynomial of much lower order than the overall transfer function. The rest of this chapter is devoted to the design of $F(z) = H(z)H(z^{-1})$ for T3 and S3.

4.4 Factorable Minimax Design Procedures

As in [29], we factor $F(z)$ as $F(z) = F_0(z)F_1^2(z)$ where $F_1^2(z)$ contains all the double zeros of $F(z)$ on the unit circle and $F_0(z)$ contains the other zeros of $F(z)$. The double zeros of $F_1^2(z)$ on the unit circle imply that it has an odd number of coefficients and that it is a zero-phase function. The zeros of $F_0(z)$ must occur in mirror image pairs reflected about the unit circle. Hence, $F_0(z)$ also has an odd number of coefficients and is a zero-phase function.

Let the lengths of $F_0(z)$ and $F_1^2(z)$ be $2l_0+1$ and $2l_1+1$ respectively. The number of coefficients of the overall Nyquist filter $F(z)$ is $M = 2(l_0 + l_1) + 1$. Note that the case $l_0 + l_1 = kK$ (for any integer k) results in a Nyquist filter with $f(-l_0 - l_1) = f(l_0 + l_1) = 0$ thereby reducing the effective length by two. The inverse z-transforms of $F(z)$, $F_0(z)$ and $F_1^2(z)$ are defined to be $f(n)$, $f_0(n)$ and $d(n)$ respectively.

4.4.1 First Method

The design procedure for the first method is as follows:

1. Initialization: Fix l_0, l_1, K, and ω_s. Set $F_0(z) = 1$. The weighting is given as $W(\omega)$.
2. Design $F_1(z)$ using the McClellan-Parks algorithm such that it has zeros only on the unit circle in the stopband region $[\omega_s, \pi]$.
3. Impose the time domain constraints by solving for the coefficients of $F_0(z)$ through a linear system of equations.
4. Form the Nyquist filter $F(z)$. If the design warrants improvement, go back to step 2.
5. Split $F(z)$ into its minimum and maximum phase parts.

We now describe steps 2 to 5 in more detail.

4.4.1.1 Step 2: Frequency Domain Specifications

The McClellan-Parks algorithm is used to get the coefficients of $F_1(z)$. The specifications are that the frequency response must be one at $\omega = 0$ and must approximate zero in the stopband region $[\omega_s, \pi]$. The weighting function applies to $F_1^2(z)$. The weighting function is $W(\omega)|F_0(e^{j\omega})|$. Initially, it is $W(\omega)$ since $F_0(z) = 1$. Subsequent iterations involve an update of the weighting function as $F_0(z)$ is recomputed. For the design of $F_1(z)$, tabulated values of the square root of the weighting function are inputs to the algorithm.

In the stopband, the frequency response of $F_1(z)$ exhibits a ripple-like behaviour with local minima and maxima occurring at the extremal frequencies. If l_1 is even, $F_1(z)$ has an odd number of coefficients (l_1+1). Two of the extremal frequencies are 0 and π [35]. However, the total number of zeros is a multiple of two, all occurring in complex conjugate pairs (no zero at $z=-1$). At $\omega=\pi$, either a local maximum or a local minimum occurs. If l_1 is odd, $F_1(z)$ has an even number of coefficients. In this case, a zero occurs at $z=-1$. However, π is not an extremal frequency. The other zeros occur in complex conjugate pairs bringing the total number of zeros to l_1.

4.4.1.2 Step 3: Time Domain Constraints

Given $F_1(z)$, we form $F_1^2(z)$ and solve for the coefficients of $F_0(z)$ such that $F(z)$ has the Nyquist property. Since $f(n)$ has samples for $n=-(l_0+l_1)$ to l_0+l_1, the number of zero-valued samples that occur as n goes from 1 to l_0+l_1 is $\lfloor (l_0+l_1)/K \rfloor$. The same holds true as n goes from -1 to $-(l_0+l_1)$. Since, the sample for $n=0$ is also known, the number of known coefficients of $F(z)$ is[†]

$$L = 2\left\lfloor \frac{l_0+l_1}{K} \right\rfloor + 1 \ . \tag{4.4}$$

The coefficients of $F(z)$ are found by performing the convolution $f_0(n) \star d(n)$. By expanding the convolution sum, one can uniquely determine $F_0(z)$ such that the time domain constraints are satisfied [29] if the number of unknown coefficients of $F_0(z)$ equals the number of known coefficients of $F(z)$. This results in a system of linear equations of dimension $2l_0+1$. By further exploiting the time domain symmetry

[†] This formula is a corrected version of the formula given in [29].

of each filter, the problem is reduced to that of a system of dimension $l_0 + 1$. The system of equations can be expressed as $\mathbf{Df} = \mathbf{c}$ where $\mathbf{f}^T = [f_0(0) \quad \cdots \quad f_0(l_0)]$, $\mathbf{c}^T = [1/K \quad 0 \quad \cdots \quad 0]$ and

$$\mathbf{D} = \begin{bmatrix} d(0) & 2d(1) & \cdots & 2d(l_0) \\ d(K) & d(K-1) + d(K+1) & \cdots & d(K-l_0) + d(K+l_0) \\ \vdots & \vdots & & \vdots \\ d(Kl_0) & d(Kl_0-1) + d(Kl_0+1) & \cdots & d(Kl_0-l_0) + d(Kl_0+l_0) \end{bmatrix} \tag{4.5}$$

The constraint that $L = 2l_0 + 1$ is equivalent to $l_0 = \lfloor (l_0 + l_1)/K \rfloor$ which in turn translates to constraints on l_0 and l_1 given by

$$l_0(K-1) \leq l_1 < l_0(K-1) + K \ . \tag{4.6}$$

Appendix E gives the derivation of closed form expressions for l_0 and l_1 in terms of K and M,

$$\begin{aligned} l_0 &= \left\lfloor \frac{M-1}{2K} \right\rfloor \\ l_1 &= \frac{M-1}{2} - \left\lfloor \frac{M-1}{2K} \right\rfloor \ . \end{aligned} \tag{4.7}$$

This method of satisfying the Nyquist property automatically takes care of the passband response of $F(z)$. Note that $F_0(z)$ is a highpass function that primarily controls the passband characteristic and hence has no zeros on the unit circle.

4.4.1.3 Step 4: Termination

The coefficients of $F(z)$ are found from $F_0(z)$ and $F_1^2(z)$. Steps 2 and 3 are iterated if the design warrants improvement. For Step 2, the weighting function $W(\omega)|F_0(e^{j\omega})|$ is updated to include a new $|F_0(e^{j\omega})|$ calculated from the coefficients of $F_0(z)$ formed in Step 3 of the previous iteration. The application of this weighting factor significantly influences the stopband behaviour of $F(z)$ through the design of

$F_1(z)$. In the weighting function, the factor $|F_0(e^{j\omega})|$ leads to a stopband behaviour of $F_1^2(z)$ that compensates for the highpass response of $F_0(z)$. The stopband behaviour of $F(z)$ is either equiripple or nonequiripple depending on the other factor $W(\omega)$ in the weighting function. The iterations are terminated when the extremal frequencies obtained by designing $F_1(z)$ do not change by more than a given threshold.

4.4.1.4 Step 5: Final Filter

This step factors $F(z)$ into minimum and maximum phase parts. The minimum phase part of $F(z)$ is $H(z) = F_0^-(z)F_1(z)$ where $F_0^-(z)$ is the minimum phase part of $F_0(z)$. The factor $F_0^-(z)$ contains all the zeros of $F_0(z)$ that are within the unit circle. The factor $F_1(z)$ (has zeros on the unit circle) is known as a byproduct of the design procedure. Only $F_0(z)$ needs to be factored in order to derive its minimum phase part. The maximum phase part, $H(z^{-1})$, is obtained by time reversing the coefficients of $H(z)$.

4.4.2 Second Method

The difference between the second method and the previous approach lies in Step 2 in which a constrained form of the McClellan-Parks algorithm is used to directly compute the coefficients of $F_1^2(z)$ rather than to first design $F_1(z)$. The specifications are that the frequency response must be one at $\omega = 0$ and must approximate zero in the stopband region $[\omega_s, \pi]$. As before, the weighting function is $W(\omega)|H_0(e^{j\omega})|$. Tabulated values of the weighting function are supplied as inputs. Since double zeros on the unit circle are required, we constrain the frequency response to be nonnegative in the stopband region. We implement the procedure in [36] (see also [37]) to obtain a minimax approximation to a desired response that satisfies given upper and lower constraints.

In the stopband, the frequency response of $F_1^2(z)$ exhibits a ripple-like behaviour with local minima and maxima occurring at the extremal frequencies. The local minima correspond to the frequencies at which the response touches zero. It is these frequencies which determine the double zeros of $F_1^2(z)$ on the unit circle. Given that $F_1^2(z)$ has $2l_1 + 1$ coefficients, a total of $l_1 + 1$ extremal frequencies result [35]. Two of the extremal frequencies are 0 and π regardless of the value of l_1. If l_1 is odd, the extremum at π is a local minimum thereby producing a double zero at $z = -1$. The other zeros occur in groups of four in the stopband region bringing the total number of zeros to $2l_1$. If l_1 is even, the extremum at π is a local maximum (no zero at $z = -1$). The total number of zeros is a multiple of four and occur in groups of four in the stopband region.

Steps 3 and 4 are identical to the first approach. In splitting $F(z)$ into its minimum and maximum phase parts, we take advantage of the fact that the frequencies corresponding to the double zeros of $F_1^2(z)$ are available as a byproduct of the modified McClellan-Parks algorithm (similar to the approach used in [37] to generate minimum phase filters). Given these frequencies and hence, the locations of the zeros on the unit circle, $F_1(z)$ can be formed without directly factoring $F_1^2(z)$. As before, only $F_0(z)$ must be factored to form $H(z) = F_0^-(z)F_1(z)$.

The next section discusses the merits of factoring only $F_0(z)$ as opposed to $F(z)$ in determining the minimum phase part. Also, observations concerning the relative orders of $F_0(z)$ and $F(z)$ are given.

4.5 The Factorization Problem

Polynomial factorization can be an ill-conditioned problem [38]. There is an advantage to substantially lowering the order of the polynomial to be factored. A general zero plot of $F(z)$ includes double order stopband zeros on the unit circle and the other simple zeros that mainly contribute to the passband response. If $F(z)$ were to be factored, the double zeros on the unit circle and the other zeros would be determined through one factorization procedure. Note that finding the double zeros can be an ill-conditioned problem [38]. Furthermore, the use of polynomial deflation can be troublesome since the zeros of the resulting polynomial may in some cases diverge from those of the original polynomial [38]. In our approach, both factorization and deflation of $F(z)$ are avoided. In particular, the knowledge of $F_1(z)$ ensures that any errors that would normally occur in locating the unit circle zeros are absent and do not affect the zeros of $F_0(z)$. Furthermore, the factorization of $F_0(z)$ does not involve multiple zeros since $F_0(z)$ has only the simple zeros of $F(z)$ that primarily influence the passband.

Since only the zeros of $F_0(z)$ have to be determined, the extent to which the factorization problem is eased depends on the ratio l_1/l_0. The ratio l_1/l_0 is both a measure of the proportion of unit circle zeros to the other zeros of $F(z)$ and of the degrees of $F(z)$ and $F_0(z)$. The higher the value of l_1/l_0, the lower the relative orders of $F_0(z)$ and $F(z)$. Appendix F shows that l_1 is greater than l_0 by a factor of at least $K-1$. Therefore, the inherent advantage in terms of polynomial factorization increases as K increases. However, even for the lowest value, $K=2$, the degree of $F(z)$ is at least twice the degree of $F_0(z)$. Note that the lower bound for $l_1/l_0 = K-1$ is satisfied when the end points of the impulse response are zero-valued (shown in Appendix F). We discard this artificial case because the values of l_0 and M can be reduced by 1 and 2 respectively thereby giving a new value of l_1/l_0.

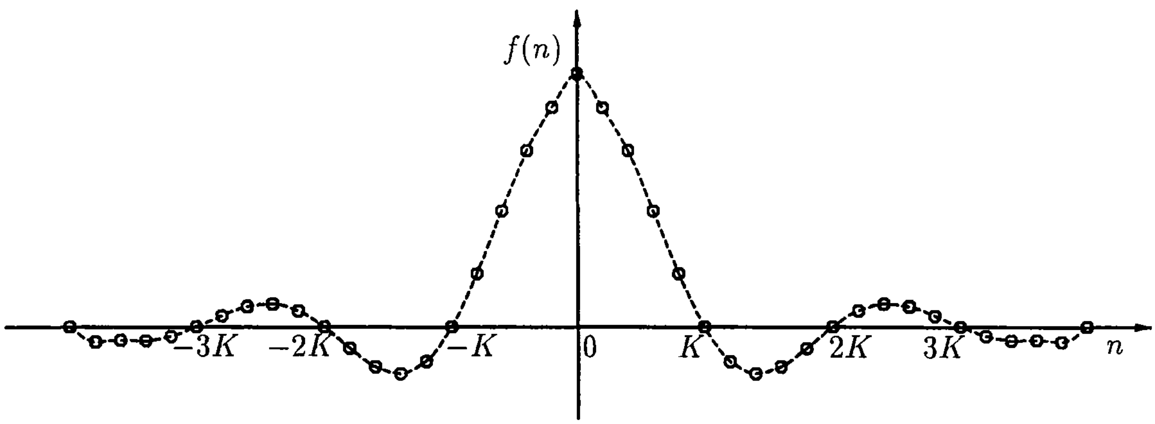


Fig. 4.1 Typical Nyquist response $f(n)$ (shown for $K=5$, $M=39$ and $\beta=0.2$)

A typical designed Nyquist response $f(n)$ is depicted in Fig. 4.1. The time response consists of a main lobe between $n=-K$ and $n=K$ and a series of sidelobes each occurring between the zero crossings. The value of l_0 is a measure of the number of sidelobes. As the number of coefficients M increases, l_1 also increases. For a fixed number of lobes (constant value of l_0), increasing M results in a higher stopband attenuation while maintaining the same factorization complexity. Hence, for a fixed number of lobes, one can maximize l_1/l_0 by increasing M. The largest disparity in the relative orders of $F_0(z)$ and $F(z)$ results by choosing the filter lengths to be of the form $M = 2kK - 1$.

Given that the filter lengths are constrained to be of the form $M = 2kK - 1$, the ratio l_1/l_0 is

$$\frac{l_1}{l_0} = \frac{k(K-1)}{k-1} \ . \tag{4.8}$$

This ratio is a maximum for $k = 2$†. As k increases, a tradeoff results in that a higher stopband attenuation due to a longer filter is obtained at the expense of both a lower l_1/l_0 and a higher l_0. The subsequent examples show that a value of $k = 5$ results in about an 80 dB stopband attenuation for a roll-off factor of 0.52. Then, $l_1/l_0 = 5(K-1)/4$ and $l_0 = 4$. Only an eighth order polynomial with simple zeros needs to be factored. Smaller roll-off factors require a larger number of taps (larger value of k) and hence, a lower value of l_1/l_0 and a higher value of l_0 for an 80 dB stopband attenuation.

4.6 Discussion of the Design Techniques

The two methods in this chapter can be used to design factorable Nyquist filters with Chebyshev stopband behaviour and exact zero crossings in its impulse response. An equiripple stopband is obtained when $W(\omega) = 1$. A nonequiripple design is achieved by specifying a nonconstant $W(\omega)$. The main advantages of the design techniques are that the polynomial factorization complexity in finding the minimum phase part is considerably eased and that arbitrary frequency weighting can be applied without additional computational overhead. This section discusses the relative merits

† This is also a unique maximum for a general M (see Appendix F).

of the two new factorable minimax methods and gives design examples. Finally, the group delay behaviour of the minimum phase part is examined.

4.6.1 Comparison of the Two Proposed Methods

In the first method, we design an unconstrained $F_1(z)$. When this $F_1(z)$ is squared, the resulting nonnegative frequency response has extremal frequencies that include those obtained in the design of $F_1(z)$. These are augmented by another set at which the response is zero. In the second method, we design $F_1^2(z)$ directly. The error is minimized over the same closed region as in the first method while maintaining the same total number of extremal frequencies. Since the constrained minimax approximation is unique [36], $F_1^2(z)$ is the same for both methods.

Despite the theoretical equivalence of the two methods, numerical differences do arise. The coefficients of $F_1^2(z)$ obtained by the two methods differ slightly in practice. Although these small differences lead to more pronounced differences in the coefficients of $F_0(z)$, the coefficients of the overall Nyquist filters formed by the two methods show only small differences. These differences manifest themselves mostly in the stopband region of the frequency response. An equiripple characteristic is more closely approached by the first method.

4.6.2 Design Examples

Examples are presented to demonstrate both equiripple and nonequiripple designs. The design computations were done using double precision floating point arithmetic. Four iterations were necessary to resolve the coefficients. The following examples are generated by the first of our methods.

Example 1

We generate an equiripple design with parameters $K = 6$, $l_0 = 4$, $l_1 = 25$, $\omega_p = 0.08\pi$ and $\omega_s = 0.254\pi$. This results in a filter with 59 coefficients having a roll-off factor $\beta = 0.52$ whose magnitude response is shown in Fig. 4.2. The passband response is flat to within 0.003 dB. The filter length is of the form $M = 2kK - 1$ with $k = 5$.

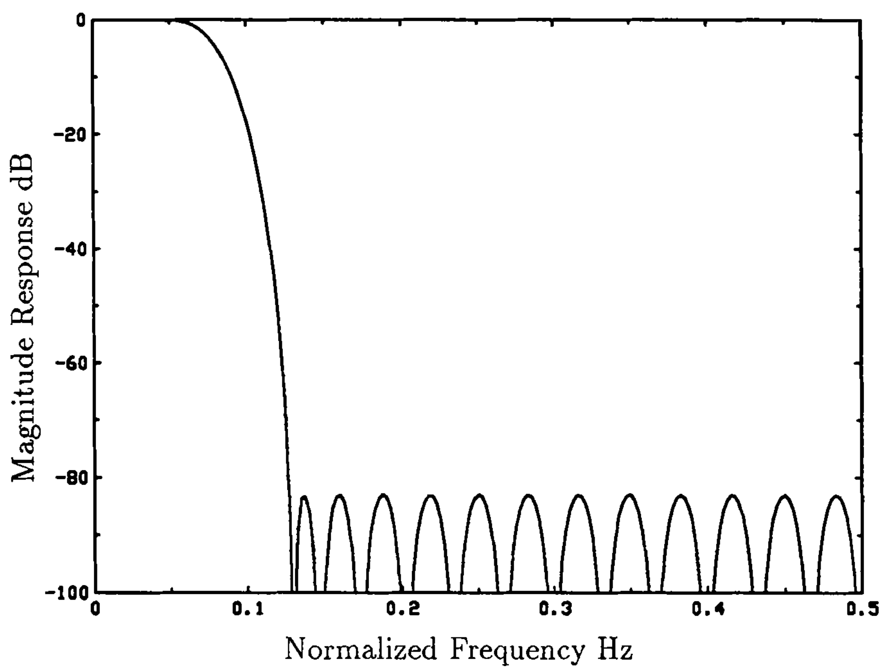


Fig. 4.2 Magnitude response of the Nyquist filter: Example 1

Example 2

The parameters used in this example are $K = 4$, $l_0 = 4$, $l_1 = 15$, $\omega_p = 0.12\pi$ and

$\omega_s = 0.38\pi$. The weighting is

$$W(\omega) = \begin{cases} 1 & \text{for } \omega = 0 \\ \dfrac{20}{2\pi}(\omega - \omega_s) + 1 & \text{for } \omega_s \leq \omega \leq \pi \end{cases} \tag{4.9}$$

This gives a nonequiripple Nyquist filter with 39 coefficients and a roll-off factor $\beta = 0.52$. The filter length is of the form $M = 2kK - 1$ with $k = 5$. Figure 4.3 shows the magnitude response of the filter. The passband response is flat to within 0.002 dB. Figure 4.4 shows the group delay response of the minimum phase part of the filter.

4.6.3 Group Delay

The group delay of the minimum phase part is only important in the passband and is primarily influenced by the passband zeros which are within the unit circle. For a given number of taps and a given K, the group delay tends to be more constant as the roll-off factor increases. Also, for a given roll-off factor and a given K, a larger number of taps produces a group delay with a greater deviation. The minimum phase filters generated in Examples 1 and 2 that achieve about a 40 dB stopband attenuation have a relatively small passband group delay variation (approximately 0.15 zero crossing intervals).

Factorization of $F(z)$ into two constant group delay functions $H(z)$ and $G(z)$ to be used in T3 and S3 is possible as follows. First, the double zeros of $F_1^2(z)$ are allocated one each to $H(z)$ and to $G(z)$. Then, we classify the zeros of $F_0(z)$ in polar form $re^{j\theta}$ and only consider $0 \leq \theta \leq \pi$. The zeros of $F_0(z)$ are taken in ascending order of θ and the mirror-image pairs are alternately assigned to $H(z)$ and $G(z)$.

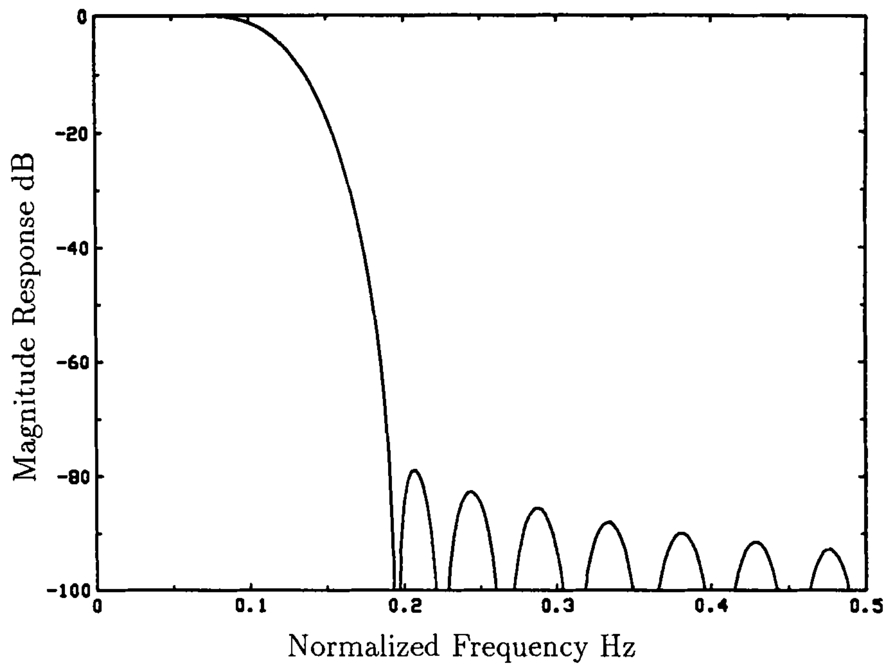


Fig. 4.3 Magnitude response of the Nyquist filter: Example 2

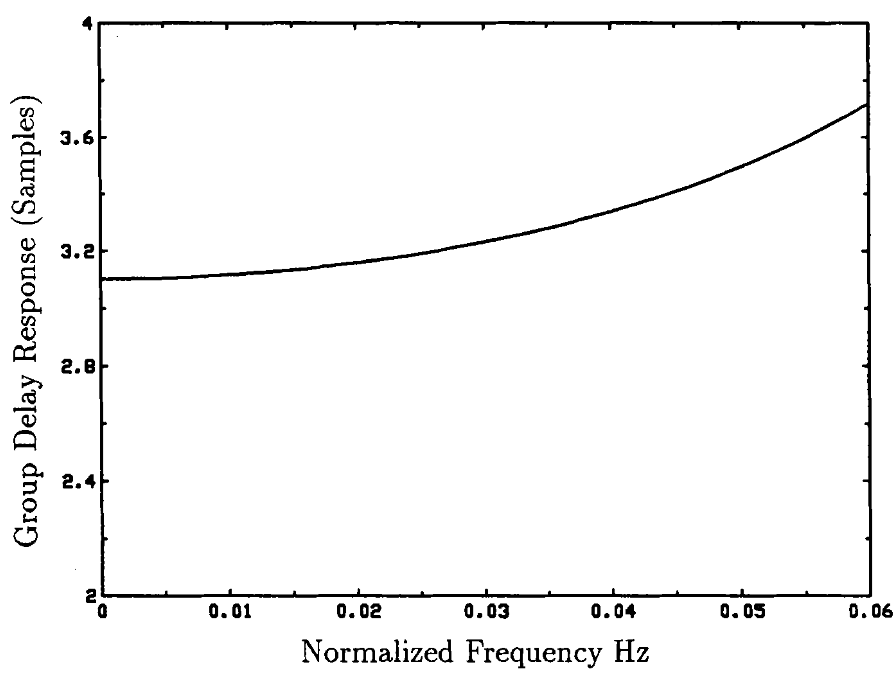


Fig. 4.4 Group delay response of the minimum phase part: Example 2

This ensures that both $H(z)$ and $G(z)$ have constant group delay. Note that if l_0 is odd, the number of taps of $H(z)$ and $G(z)$ differ by two. Otherwise, they have the same number of taps. Due to the presence of identical stopband zeros in $H(z)$, $G(z)$ and $F(z)$, the stopband responses of both $H(z)$ and $G(z)$ are good. However, the passband responses can deviate significantly from a constant. Therefore, factorizing a Nyquist filter into two general factors $H(z)$ and $G(z)$ for use in T3 and S3 will assure constant group delay factors but at the expense of an acceptable magnitude response.

4.7 Comparison With Other Approaches

This section discusses the relative merits of the factorable minimax design methods when compared with other approaches.

4.7.1 Linear Programming Technique

In [27], a linear programming approach that is also based on a minimax criterion is used to design a factorable Nyquist filter. For comparison, we generate a filter with the same parameters as the example in [27] ($M = 31$, $K = 4$, $\beta = 0.125$ and $W(\omega) = 1$) using our factorable minimax approach. It is observed that the magnitude and group delay responses of the filters given by the two designs are very similar. The equiripple magnitude characteristic is more exactly given by our approach. Arbitrary weighting can be easily applied in both the factorable minimax approach and a linear programming formulation (see [26]).

4.7.2 Eigenfilter Formulation

The eigenfilter approach [29] also simplifies the factorization problem and meets the time domain constraints by solving a linear system of equations. The differences between the factorable minimax approach and the eigenfilter method are as follows. First, our approach is based on a minimax criterion as opposed to a least squares design achieved by the eigenfilter method. The factorable minimax approach naturally generates an equiripple behaviour whereas the eigenfilter method naturally renders nonequiripple filters. However, weighting can be applied in both methods to alter the stopband characteristic. For the factorable minimax method, the McClellan-Parks algorithm can easily incorporate arbitrary weighting, whereas, the incorporation of an arbitrary weighting factor into the eigenfilter formulation involves the use of numerical integration techniques.

A design example illustrates the differences in performance of the two methods. Identical parameters to the ones in [29] are used. In particular, $K = 3$, $l_0 = 10$, $l_1 = 21$, $\omega_p = 0.233\pi$, $\omega_s = 0.433\pi$ and $W(\omega) = 1$. This gives a Nyquist filter with 63 coefficients and a roll-off factor $\beta = 0.3$. Figure 4.5 shows the magnitude response of the minimum phase part generated by our factorable minimax method. The stopband attenuation of the minimum phase filter achieved by our method is about 48 dB whereas the first stopband ripple of its counterpart generated by the eigenfilter method shows an attenuation of approximately 45 dB. For higher frequencies, the ripples of the filter designed by the eigenfilter method show an attenuation that is more than that achieved by our method.

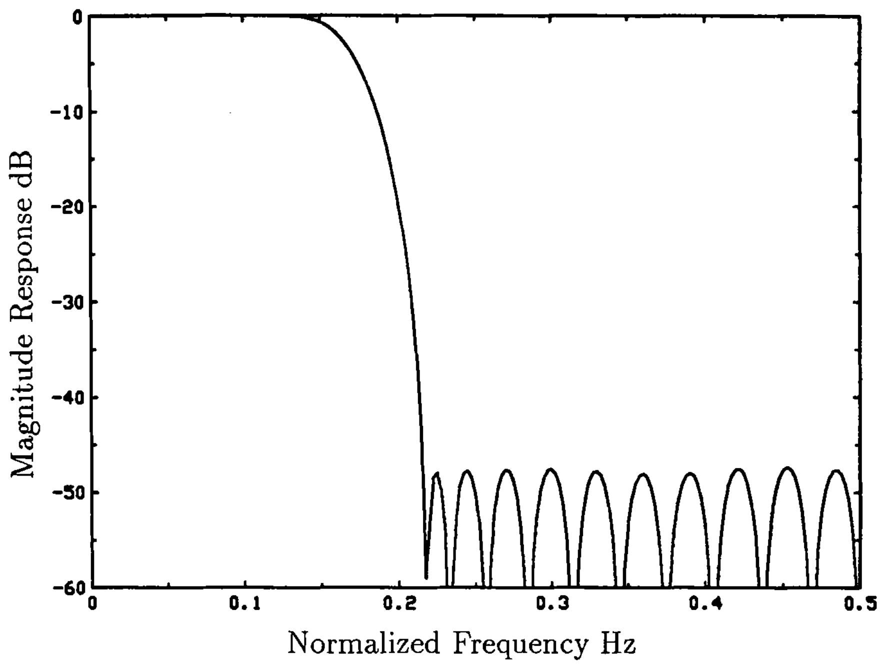


Fig. 4.5 Magnitude response of the minimum phase part of the Nyquist filter designed with the same parameters as in [29]

4.7.3 Direct Use of the McClellan-Parks Algorithm

Factorable Nyquist filters can also be designed by invoking the constrained form of the McClellan-Parks algorithm [36] to get a nonnegative response that approximates a raised cosine characteristic. This approach and our factorable minimax method can be used for getting the prototypes for systems T3 and S3. We compare the two methods from different points of view (stopband attenuation, group delay, factorization problem and achievement of exact zero crossings) through a design example that conforms to the CCITT recommendation V.22 [39].

The CCITT recommendation V.22 [39] includes the specification of a pair of transmitter/receiver filters which should approximate the square root of a raised cosine response. The specified roll-off factor is 0.75. Upper and lower bounds in the frequency response in both the passband, transition band and a small portion of the stopband must be met. In addition, the group delay variation should be below a prescribed limit in the passband and a portion of the transition band.

We design Nyquist filters with a roll-off factor of 0.75 and with $K = 4$ by the approach that uses the McClellan-Parks algorithm and the first factorable minimax method. The approaches are described in slightly more detail as follows:

1. Design a filter that approximates a raised cosine response by invoking the constrained form of the McClellan-Parks algorithm [36] such that the response is nonnegative and its minimum and maximum phase parts have a frequency response that satisfies the upper and lower bounds specified by V.22.
2. Use the first factorable minimax method to design a Nyquist filter such that its minimum and maximum phase parts satisfy the V.22 specifications of the frequency response.

In all cases, the smallest number of taps that satisfy the constraint $M = 2kK - 1$ is used. This leads to 15 tap Nyquist filters for the two methods. A constant weighting of 1 is used in both cases thereby yielding equiripple behaviour.

Factorable Nyquist filters designed by Method 1 can be made to satisfy the magnitude specifications of V.22 since the procedure in [36] takes upper and lower bounds of the frequency response into account. However, there is no guarantee that the group delay variation of the minimum phase part is assured to be below the required limit. The factorable minimax method does not guarantee a filter that satisfies any prescribed specifications of the frequency response. However, filters that satisfy the V.22 specifications can be designed by choosing the number of taps, carrying out the design and finally verifying that the constraints are met. We find that the constraints are met with 15 taps. It is observed that increasing the number of taps will cause the frequency response constraints to be violated since the transition band becomes more steep and lies outside the acceptable region.

For performing a min/max phase split, factoring $F(z)$ designed by Method 1 can be avoided since the unit circle zeros can be extracted from the extremal frequencies. However, the other zeros would have to be determined by first deflating the original polynomial. Also, there is no general expression for the proportion of unit circle zeros to the other zeros of $F(z)$. It is observed in [38] that deflation is more stable if the zeros of smaller magnitude were extracted first. This further discourages the division of the original polynomial by a polynomial that has the unit circle zeros since they have a larger magnitude than the zeros within the unit circle which should be extracted first to enhance the stability of the deflation process. A remedy to this problem is to use Lagrange interpolation as in [37] to obtain a polynomial that represents the passband zeros and then factor it to obtain the zeros inside the unit circle. An alternative is to use a modified Newton's iteration [40] on the original polynomial to obtain the zeros inside the unit circle. Method 2 directly separates $F(z)$ into two polynomials $F_1^2(z)$ and $F_0(z)$ having zeros on and off the unit circle respectively. This avoids the tasks of approximating $F_0(z)$ by Lagrange interpolation and determining the zeros of $F_0(z)$ by considering the original $F(z)$.

After carrying out the design of the 15 tap Nyquist filters by both Methods 1 and 2, we compare them in terms of the stopband attenuation achieved by $F(z)$, the group delay of the factorized minimum phase filter $H(z)$ in the region considered in the V.22 specifications and in terms of the residual intersymbol interference. Method 1 does not assure exact zero crossings in the time response $f(n)$. Hence, we use two quantitative measures of the residual intersymbol interference to measure the suppression. Specifically, the normalized peak distortion D_{P} and the normalized RMS distortion D_{RMS} are computed. They are defined by

$$D_{\mathrm{P}} = \frac{\sum\limits_{\substack{n \\ n\neq 0}} |f(nK)|}{|f(0)|} \tag{4.10}$$

and

$$D_{\mathrm{RMS}} = \sqrt{\frac{\sum\limits_{\substack{n \\ n\neq 0}} f^2(nK)}{f^2(0)}} \; . \tag{4.11}$$

The stopband attenuations of $F(z)$ achieved by Methods 1 and 2 are about 45 and 50 dB respectively. The allowable variation in group delay as specified by V.22 is 0.18 zero crossing intervals. Method 1 generates a minimum phase filter whose group delay variation is slightly under the prescribed 0.18 zero crossing intervals. Method 2 does not meet the group delay requirement in that the filter it produces has a variation of 0.24 zero crossing intervals†. In terms of normalized peak and RMS distortion, Method 2 assures exact zero crossings and hence, produces no such distortion. Method 1 produces low distortions $D_{\mathrm{P}} = 0.0004$ and $D_{\mathrm{RMS}} = 0.0003$. Method 2 gives a higher stopband attenuation than Method 1 and produces exact zero crossings in the impulse response. This enhanced stopband attenuation comes at the expense of a larger group delay variation.

A comparison of the factorable minimax method to an approach directly using the McClellan-Parks algorithm in terms of satisfying a CCITT recommendation was done. Concerning the design of Nyquist filters for T3 and S3, the new factorable minimax

† A simple second order allpass equalizer brings the group delay within specifications. However, the use of such equalizers sacrifices the exact zero crossing property of the original design.

method does offer advantages over its McClellan-Parks counterpart. First, the new method leads to exact zero crossings in the impulse response. The factorization problem can be alleviated in both approaches. However, the new method can bring down the factorization complexity by choosing appropriate filter lengths. Also, the polynomial representing the zeros off the unit circle is directly computed in the new method. Hence, this does not necessitate any polynomial approximation or a zero finding algorithm based on the original $F(z)$.

Chapter 5 Optimized Filter Banks

The minimax design procedures described in the previous chapter give lowpass prototypes such that the Nyquist criterion is either approximated or exactly satisfied. The designs are based on a common input-output transfer function for every pair of terminals in the transmultiplexers. Returning to the synthesis procedure in Chapter 3, we note that the achievement of a common input-output transfer function partially relies on the bandlimitedness of the prototype. Moreover, the crosstalk-free nature is heavily dependent on the bandlimitedness property in that this property is used to cancel the crosstalk terms (which comprise the crosstalk functions) that do not involve spectral overlap. As before, we refer to bandlimited lowpass prototypes as those with a stopband response which is exactly zero. Since bandlimited prototypes cannot be designed, there exist practical imperfections in the synthesized systems. First, the input-output transfer function may be different for each terminal pair. Second, there may be residual crosstalk between signals sent at non-adjacent center frequencies.

We proceed to analyze the synthesized transmultiplexers with respect to both the input-output transfer function and the crosstalk when practical filters are used. A practical lowpass prototype is not bandlimited in that its frequency response only approximates zero in the stopband (stopband attenuation is high but not infinite). Based on the analysis, optimized lowpass FIR filters that attempt to achieve a high suppression of both intersymbol interference and crosstalk are designed by minimizing an error function. Therefore, the practical degradations (both intersymbol interference and crosstalk) are taken into account in the filter design procedure. In contrast to the minimax approaches, the optimized designs take crosstalk into account. The performance of the transmultiplexers is evaluated with both the optimized and the minimax filters. We compare the two design approaches with respect to the resulting performance. Finally, the feasibility of this technique for the subband complements is discussed.

5.1 System Imperfections

Transmultiplexers T1 through T5 have each been configured with bandlimited filters such that (1) the input-output transfer function is the same for every pair of corresponding terminals and (2) crosstalk is cancelled. In addition, satisfying the Nyquist criterion eliminates intersymbol interference and hence, achieves perfect reconstruction. With practical filters, the input-output transfer function may not be the same for all pairs of terminals. In addition, the design procedure may give filters such that the Nyquist criterion is not exactly satisfied. Therefore, intersymbol interference need not be eliminated at each output terminal. Moreover, the use of practical filters may lead to residual crosstalk which would otherwise be cancelled with a bandlimited prototype.

In this section, we further analyze each transmultiplexer in terms of the possible limitation of not achieving perfect reconstruction due to the use of practical filters. The next section shows how the limitation is taken into account in an optimized design of the practical prototype. Then, the performance of the systems with practical filters is evaluated.

5.1.1 The Input-Output Transfer Function

In analyzing the transmultiplexers, we return to the synthesis procedure in Chapter 3 to see where the bandlimitedness property was used in getting a common input-output transfer function. Consider the general expression for the input-output transfer function $T_{kk}(z^N)$ given by Eq. (3.5). The bandlimitedness of the lowpass prototype was invoked to cancel the last term $\sum_{i=0}^{N-1} W^{i(n_k-p_k)} 2\cos(\alpha_k - \beta_k) H(e^{-j\omega_k} zW^{-i}) H(e^{j\omega_k} zW^{-i})$ for some of the terminals. However, this term is naturally cancelled for all terminals in T2, T4 and T5 and for the terminals in T1 operating at the center frequencies of 0 and π. Similarly, for system T3, the general expression for the input-output transfer function $T_{kk}(z^N)$ is examined. The bandlimitedness of the prototypes must be invoked to cancel some terms in $T_{kk}(z^N)$ for the terminals that do not operate at the end frequencies. These terms are naturally cancelled (without invoking the bandlimitedness property) for the terminals operating at the end frequencies of 0 and π.

The preceding analysis reveals that the input-output transfer function is indeed the same for all pairs of terminals in each of the systems T2, T4 and T5. Moreover, this property holds for any practical prototype $H(z)$. Therefore, for any $H(z)$, the common input-output transfer function $T(z^N)$ is given by Eq. (3.17) for system T2 and by Eq. (3.21) for T4 and T5. Now, consider systems T1 and T3. The common input-output transfer function $T(z^N)$ as given in Eq. (3.15) (system T1) and in Eq. (3.19) (system T3) holds only for the terminals specified by center frequencies of 0 and π. Again, this is true for practical prototypes. The input-output transfer functions for the other terminals of T1 and T3 are different from those given by Eqs (3.15) and (3.19) when practical filters are used. These differences are due to the fact that the prototypes are not bandlimited.

The next step is to identify the sources of intersymbol interference in each of the transmultiplexers. In systems T2, T4 and T5, intersymbol interference is cancelled at all terminals given any $H(z)$ if $H^2(z)$ satisfies the Nyquist criterion. The only potential source of intersymbol interference is due to the limitation of the design procedure in giving $H(z)$ such that $H^2(z)$ does not exactly satisfy the Nyquist criterion. Therefore, the minimax design of Chapter 4 will lead to residual intersymbol interference in T2, T4 and T5.

When dealing with systems T1 and T3, two cases must be considered. First, consider the terminals operating at center frequencies of 0 and π. At these terminals, the only source of intersymbol interference is due to the design procedure in giving filters such that the Nyquist criterion is not exactly satisfied. At the other terminals, an additional source of intersymbol interference arises since the filters are not bandlimited. Given the minimax design of the previous chapter, intersymbol interference will be present at all the terminals of T1. For transmultiplexer T3 with $G(z) = H(z^{-1})$, the factorable minimax design method assures that no intersymbol interference is present at the terminals specified by center frequencies of 0 and π. However, intersymbol interference distortion will exist at the other terminals of T3 since the prototypes are not bandlimited.

5.1.2 Crosstalk Functions

Here, we wish to determine the sources of crosstalk that arise with practical filters. From the synthesis procedure in Chapter 3, crosstalk cancellation with bandlimited prototypes occurs in two ways. First, terms in the crosstalk function $T_{kl}(z^N)$ that involve either partial or complete spectral overlap are cancelled by choosing the center frequencies, delays and phases. This cancellation depends only on the center frequencies, delays and phases and is independent of any particular form of $H(z)$ and $G(z)$. Therefore, these terms continue to be cancelled with practical filters. Second, terms in the crosstalk function that do not involve spectral overlap are zero due to the bandlimitedness of the prototypes. However, these crosstalk terms are not zero with practical filters. This will lead to residual crosstalk. Summarizing, we note that all the crosstalk terms in $T_{kl}(z^N)$ that involve spectral overlap with bandlimited filters continue to be cancelled with practical filters.

Note that with practical filters, although the terms in $T_{kl}(z^N)$ that involve spectral overlap are cancelled (as discussed above), this does not generally imply that $T_{kl}(z^N) = 0$. We further analyze each of the transmultiplexers to determine the number of crosstalk functions that are exactly zero with practical filters (also referred to in the sequel as exact crosstalk cancellation). Exact crosstalk cancellation depends only on the center frequencies, delays and phases and occurs independently of the prototypes $H(z)$ and $G(z)$. For a particular output terminal, there are $N-1$ crosstalk functions. For each of the transmultiplexers, a certain number of these $N-1$ functions may be exactly zero. We proceed to enumerate the number of exact crosstalk cancellations.

In system T1, the crosstalk is exactly zero between two signals sent at the same center frequency, at center frequencies separated by an odd multiple of $2\pi/N$ and at center frequencies separated by an even multiple of $2\pi/N$ if the difference in the delay factors is an odd multiple of $N/2$. In system T2, exact crosstalk cancellation occurs between any two signals as long as the difference in the delay factors of the associated combining and separation filters is an odd multiple of $N/2$. System T3, like T1, has crosstalk functions involving one prototype for signals sent at center frequencies separated by an odd multiple of $2\pi/N$. For these cases, the crosstalk function is exactly zero. When two prototypes are involved in the crosstalk function, exact crosstalk cancellation only occurs between two signals sent with a center frequency of $\pi/2$ (this center frequency appears when N is a multiple of 4). For transmultiplexer T4, none of the crosstalk functions is exactly zero. In T5, the crosstalk function $T_{kl}(z^N)$ is exactly zero if $k+l=N-1$ for N not a multiple of 4. If N is a multiple of 4, $T_{kl}(z^N)$ is never exactly zero in T5.

Given the preceding discussion, all the cases were examined in detail and the number of exact crosstalk cancellations enumerated for each output terminal. Table 5.1 summarizes the results. Appendix G gives the derivation of one case for system T1, namely, for output terminals operating at center frequencies that are even multiples

Transmultiplexer	Number of Cancellations
T1	$\approx \frac{3N}{4}$
T2	$\frac{N}{2}$
T3	$\approx \frac{N}{2}$
T4	0
T5	0 or 1

Table 5.1 Number of Exact Crosstalk Cancellations for Each Output Terminal

of $2\pi/N$ when N is a multiple of 4. We see that for the case explored in Appendix G, the number of exact crosstalk cancellations is different for the two terminals at each of these center frequencies. At one of the terminals, there are $(3N-4)/4$ exact crosstalk cancellations. At the other terminal, $(3N+4)/4$ exact crosstalk cancellations occur. A similar situation in T1 develops when N is not a multiple of 4 and the center frequencies are either even or odd multiples of $2\pi/N$. In this case, the two terminals at these frequencies will show a different number of crosstalk functions that are exactly zero. The number of exact crosstalk cancellations is approximately $3N/4$ for all the terminals.

Transmultiplexer T3 has approximately $N/2$ exact crosstalk cancellations at each output terminal, the actual number depending on whether a center frequency of $\pi/2$ is used. Transmultiplexers T2 and T4 have $N/2$ and 0 exact crosstalk cancellations at each output terminal respectively. In system T5, one crosstalk function is exactly zero for each output terminal when N is not a multiple of 4. When N is a multiple of 4, none of the crosstalk functions is exactly zero in T5.

Of the transmultiplexers, T1 achieves the most number of exact crosstalk cancellations (about 3/4 of the total number of crosstalk functions). In systems T2 and T3, about half of the crosstalk functions are exactly zero. The table shows that for reasonably large N, the QAM schemes (T1, T2 and T3) achieve many more exact crosstalk cancellations than their VSB counterparts (T4 and T5).

5.2 Error Function Formulation

This section discusses the design of an FIR lowpass prototype that is based on the minimization of an error function. We consider both the practical degradations of intersymbol interference and crosstalk in the design of the prototype. The minimax filter design approaches of Chapter 4 are based on the input-output transfer functions given in Chapter 3 (common for all terminals). Also, the crosstalk is not explicitly considered.

We establish an error function that takes the various distortions into account. Minimizing the error function should give a lowpass prototype with a good stopband behaviour and in addition, should lead to low intersymbol interference and crosstalk distortions. As for the minimax design, the stopband edge frequency is $\omega_s = (1 + \beta)\omega_{\text{min}}$ where ω_{min} is the minimum bandwidth of the lowpass prototype and $0 \leq \beta \leq 1$. We recall that $\omega_{\text{min}} = \pi/N$ for T1, T2 and T3 and $\omega_{\text{min}} = \pi/2N$ for T4 and T5. Also, the parameter β (introduced in Chapter 4) is the roll-off factor that controls the bandwidth in excess of $\omega_{\min}$. Note that the passband characteristic is not explicitly considered since an approximately zero stopband response and a low intersymbol interference distortion ensure an approximately constant passband response if $\beta < 1$.

A linear phase prototype $h(n)$ is designed for systems T1, T2, T4 and T5. For notational convenience, we assume throughout that $h(n)$ is in zero-phase form and has $2L+1$ taps from $n = -L$ to L. A nonlinear phase $h(n)$ with $L+1$ taps from $n = 0$ to L is designed for transmultiplexer T3 with $G(z) = H(z^{-1})$. The error function is a weighted linear combination of various factors, each of which is discussed below.

Stopband

The factor in the error function representing the stopband characteristic is denoted by E_{sb} where

$$\sqrt{E_{\text{sb}}} = \frac{1}{2\pi} \int_S |H(e^{j\omega})|^2 d\omega \ , \tag{5.1}$$

$S = [-\pi, -\omega_s] \bigcup [\omega_s, \pi]$ and ω_s is the stopband edge. Therefore, E_{sb} is the square of the energy in the stopband. This function has been used in [28] as part of a general least-squares linear phase FIR design. For a zero-phase $H(z)$ with an odd number of taps (designed for the one prototype systems), the frequency response can be expressed as

$$H(e^{j\omega}) = \sum_{n=0}^{L} b(n) \cos \omega n \tag{5.2}$$

where $b(0) = h(0)$ and $b(n) = 2h(n)$ for $n \neq 0$. The quantity $\sqrt{E_{\text{sb}}}$ can be expressed as $\mathbf{b}^T \mathbf{P} \mathbf{b}$ where $\mathbf{b} = [b(0)\ b(1)\ \cdots\ b(L)]^T$ and $\mathbf{P}$ is a positive definite symmetric matrix whose entries are given by

$$P(r,s) = \frac{1}{\pi}\int_{\omega_s}^{\pi} \cos r\omega \cos s\omega d\omega \tag{5.3}$$

for $0 \le r, s \le L$.

Since $G(z) = H(z^{-1})$ in system T3, the stopband energies of both filters are the same. For a nonlinear phase $H(z)$, $\sqrt{E_{\text{sb}}}$ can again be expressed in quadratic form $\mathbf{h}^T\mathbf{R}\mathbf{h}$ where $\mathbf{h} = [h(0)\ h(1)\ \cdots\ h(L)]^T$ and $\mathbf{R}$ is a positive definite symmetric matrix whose entries are given by

$$\begin{aligned} R(r,s) &= \frac{1}{2\pi}\int_S e^{j\omega(r-s)} d\omega \\ &= \frac{1}{\pi}\int_{\omega_s}^{\pi} \cos(\omega(r-s)) d\omega \end{aligned} \tag{5.4}$$

for $0 \le r, s \le L$.

Intersymbol Interference Distortion

At output terminal l, the mean-square intersymbol interference distortion is given by $\frac{1}{N^2}\sum_{n\neq 0} t_{ll}^2(n)$ where $t_{ll}(n)$ is the inverse z-transform of the input-output transfer function $T_{ll}(z)$. The mean-square intersymbol interference distortion depends on which output terminal is considered. However, given the discussion in Section 5.1.1, the transfer function is the same for many input-output terminal pairs when practical filters are used. Therefore, the mean-square intersymbol interference distortion will be the same at many output terminals.

Consider systems T2, T4 and T5. As mentioned in Section 5.1.1, $t_{ll}(n)$ is the same for every terminal l even with practical filters. Hence, it is sufficient to determine the mean-square intersymbol interference distortion at only one terminal. Moreover,

$t_{ll}(n)$ is the inverse z-transform of $T(z)$ where $T(z^N)$ is defined in Eq. (3.17) for T2 and Eq. (3.21) for T4 and T5. Therefore, the mean-square intersymbol interference distortion is $\frac{1}{4}\sum_{n\neq 0} v^2(nN)$ for T2 and $\frac{1}{4}\sum_{n\neq 0} v^2(2nN)$ for T4 and T5 where $v(n) = h(n) * h(n)$ ($*$ is the convolution operator).

In systems T1 and T3, $t_{ll}(n)$ is generally different for each terminal l with practical filters. As mentioned in Section 5.1.1, these differences are due to the fact that the prototypes are not bandlimited. We ignore the differences in $t_{ll}(n)$ and only consider the terminal at either a center frequency of 0 or π. At each of these terminals, $t_{ll}(n)$ is the inverse z-transform of $T(z)$ where $T(z^N)$ is defined in Eq. (3.15) for T1 and in Eq. (3.19) for T3. Therefore, the mean-square intersymbol interference distortion at each of these terminals is $\frac{1}{4}\sum_{n\neq 0} v^2(nN)$ for T1 and $\frac{1}{4}\sum_{n\neq 0} w^2(nN)$ for T3 where $v(n) = h(n) * h(n)$ and $w(n) = h(n) * h(-n)$.

The factor representing the mean-square intersymbol interference distortion is denoted by E_{isi}. For systems T2, T4 and T5, E_{isi} is based on any terminal l. However, for T1 and T3, E_{isi} is based on the terminal at either a center frequency of 0 or π. From the preceding discussion, E_{isi} is given by

$$E_{\text{isi}} = \begin{cases} \displaystyle\sum_{\substack{n=cN \\ n\neq 0}} [h(n) * h(n)]^2 & \text{for systems T1 and T2} \\ \displaystyle\sum_{\substack{n=2cN \\ n\neq 0}} [h(n) * h(n)]^2 & \text{for systems T4 and T5} \\ \displaystyle\sum_{\substack{n=cN \\ n\neq 0}} [h(n) * h(-n)]^2 & \text{for system T3} \end{cases} \tag{5.5}$$

Note that E_{isi} is a function of $\mathbf{b}$ for the one prototype systems and is a function of $\mathbf{h}$ for T3.

Crosstalk Distortion

At output terminal l, the total crosstalk power due to the undesired input signals is $P_{\text{ctk}}(l)$. In developing a mathematical formula for $P_{\text{ctk}}(l)$, we assume that each of the input data signals is zero-mean, white, uncorrelated with other inputs and has a signal power P_{s}. The crosstalk power at output terminal l contributed by a signal at input terminal k is given by the input signal power P_{s} multiplied by $\frac{1}{N^2}\sum_n t_{kl}^2(n)$ where $t_{kl}(n)$ is the inverse z-transform of the crosstalk function $T_{kl}(z)$. Also, the total crosstalk power at output terminal l is the sum of the crosstalk powers contributed by each of the undesired signals and is given by

$$P_{\text{ctk}}(l) = \frac{P_{\text{s}}}{N^2} \sum_{\substack{k=0 \\ k\neq l}}^{N-1} \sum_n t_{kl}^2(n) \,. \tag{5.6}$$

To include the crosstalk power for every terminal l, we formulate an overall crosstalk factor E_{ctk} given by

$$\begin{aligned} E_{\text{ctk}} &= \frac{1}{P_{\text{s}}} \sum_{l=0}^{N-1} P_{ctk}(l) \\ &= \frac{1}{N^2} \sum_{l=0}^{N-1} \sum_{\substack{k=0 \\ k\neq l}}^{N-1} \sum_n t_{kl}^2(n) \\ &= \sum_{l=0}^{N-1} \sum_{\substack{k=0 \\ k\neq l}}^{N-1} \sum_{n=cN} [a_k(n) * b_l(n)]^2 \end{aligned} \tag{5.7}$$

Recall that $a_k(n)$ and $b_l(n)$ are the impulse responses of the kth combining filter and the lth separation filter respectively. Note that E_{ctk} is a function of $\mathbf{b}$ for the one prototype systems and a function of $\mathbf{h}$ for T3.

For computational purposes, the number of terms involved in the expression for E_{ctk} can be decreased by exploiting the symmetry of the crosstalk power and the fact that there may be some crosstalk functions that are exactly zero. The total crosstalk power for output terminal l operating at a center frequency ω_l is the same as that for a terminal operating at $\pi - \omega_l$ (except for $\omega_l = \pi/2$ in some systems). Hence, only the output terminals operating at frequencies in the range $[0, \pi/2]$ need be considered. After taking advantage of the symmetry described above, we can further exclude the terms in E_{ctk} corresponding to the crosstalk functions which are exactly zero.

Overall Error Function

The overall error function to be minimized is the weighted sum of the individual factors relating to the stopband, mean-square intersymbol interference distortion and total crosstalk power. At this point, note that the zero solution ($\mathbf{b} = 0$ or $\mathbf{h} = 0$) is the global minimum. To avoid reaching this solution, we append a term $(\mathbf{b}^T\mathbf{b} - 1)^2$ or $(\mathbf{h}^T\mathbf{h} - 1)^2$ to the overall error function. Hence, the overall error function $E(\mathbf{b})$ (applies to T1, T2, T4 and T5) and $E(\mathbf{h})$ (applies to T3) are

$$
\begin{aligned}
E(\mathbf{b}) &= \gamma_1 E_{\text{sb}} + \gamma_2 E_{\text{isi}} + \gamma_3 E_{\text{ctk}} + \gamma_4 (\mathbf{b}^T\mathbf{b} - 1)^2 \\
E(\mathbf{h}) &= \gamma_1 E_{\text{sb}} + \gamma_2 E_{\text{isi}} + \gamma_3 E_{\text{ctk}} + \gamma_4 (\mathbf{h}^T\mathbf{h} - 1)^2
\end{aligned}
\tag{5.8}
$$

where the γ_i represent nonnegative weighting factors. With $\gamma_3 = 0$ (no crosstalk factor), the same $E(\mathbf{b})$ and hence, the same filter results for systems T1 and T2 and for T4 and T5.

Optimization Procedure

We use a Quasi-Newton approach [41] to get a local minimum of E. It is an iterative method specified by the two equations,

$$\begin{aligned} \mathbf{H}_k \mathbf{s}_k &= -\nabla \mathbf{E}(\mathbf{d}_k) \\ \mathbf{d}_{k+1} &= \mathbf{d}_k + \lambda_k \mathbf{s}_k \end{aligned} \tag{5.9}$$

where k is the iteration index, $\mathbf{H}_k$ is the Hessian matrix, $\mathbf{s}_k$ is the direction of descent, $\nabla\mathbf{E}$ is the gradient of E and λ_k is a scaling factor which specifies the extent to which movement along the direction of descent occurs to get an update. Note that $\mathbf{d}$ is the vector of variables to be optimized and is updated in each iteration. Then, $\mathbf{d} = \mathbf{b}$ for the one prototype systems and $\mathbf{d} = \mathbf{h}$ for T3. We express the gradient $\nabla\mathbf{E}$ in closed form and evaluate it at $\mathbf{d}_k$ in each iteration. Although the Hessian matrix can be expressed in closed form, we use the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update [41] in each iteration. In the actual implementation, we use a routine from the IMSL library [42] to perform the minimization. An initial condition is supplied as an input. Also, subroutines to calculate the error function and its gradient are supplied by the user.

5.3 Design Examples

When performing an unconstrained minimization of the error function, we use the optimization procedure described above. The computations were performed using double precision floating point arithmetic. Note that the initial conditions affect the final local minimum. For the one prototype systems, the initial condition we use corresponds to an equiripple linear phase filter (with unity gain at zero frequency) having a frequency response that is a minimax approximation of the square root of a raised cosine spectrum. For transmultiplexer T3, the initial condition we use corresponds to an equiripple minimum phase filter (with unity gain at zero frequency) that is designed by the factorable minimax approach. Examples of magnitude response plots are shown in Figure 5.1 (system T1), Figure 5.2 (system T3) and Figure 5.3(a) and (b) (system T4) for the case $N = 6$ and $\beta = 0.52$. Figure 5.1 shows the magnitude response of a 33 tap filter designed with weighting factors $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$. Figure 5.2 shows the magnitude response of a 30 tap filter designed with weighting factors $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$. Figure 5.3 shows the magnitude responses of a 59 tap filter designed with weighting factors $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 0, 0.01)$ and $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$. Note that the magnitude response in the passband is flat to within 0.013 dB (Fig. 5.1), 0.003 dB (Fig. 5.2) and 0.014 dB (Fig. 5.3(a) and (b)).

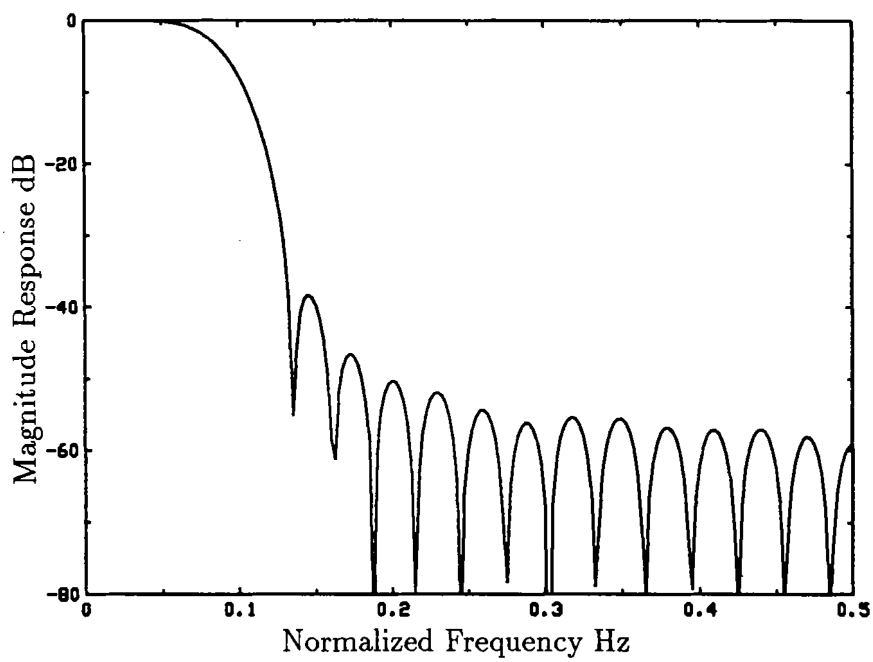


Fig. 5.1 Magnitude response of the lowpass filter for system T1. The weighting factors are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$.

The fact that some crosstalk terms which form the crosstalk function $T_{kl}(z^N)$ are exactly zero is reflected in the frequency response of the lowpass prototype. Consider Fig. 5.3 which shows the magnitude responses of the optimized filters for system T4 with and without a crosstalk weight γ_3. The stopband response is significantly different for the two filters. When a positive crosstalk weight is applied, the stopband response is shaped so as to suppress the nonzero crosstalk terms. An analysis of system T4 revealed that none of the crosstalk functions $T_{kl}(z^N)$ is exactly zero. However, some of the terms in the crosstalk function $T_{kl}(z^N)$ are zero. Among the crosstalk functions in T4 for the case $N = 6$, the terms involving sidebands whose center frequencies are separated by $\pi/3$, $2\pi/3$ and π are never zero. The other terms involving sidebands whose center frequencies are separated by $\pi/6$, $\pi/2$ and $5\pi/6$ are consistently zero. This manifests itself in that the magnitude response in the stopband around the frequencies of $\pi/3$, $2\pi/3$ and π exhibit a higher attenuation than neighbouring regions. It is the higher attenuation in these regions that suppress the nonzero crosstalk terms. Similarly, transmultiplexer T3 has nonzero crosstalk terms involving sidebands separated by $2\pi/3$ when $N = 6$. When the crosstalk weight $\gamma_3 = 1$, the stopband response of the resulting filter is better than for a design with $\gamma_3 = 0$ about the frequency $2\pi/3$.

Additional experiments were conducted by changing only the parameter γ_4 (the weighting factor for the term that avoids a zero solution) and observing the performance in terms of intersymbol interference and crosstalk distortions. The value $\gamma_4 = 0.01$ was chosen to arrive at a good solution in a reasonable number of iterations. Reducing γ_4 significantly below this value gives a local minimum with a poorer per-

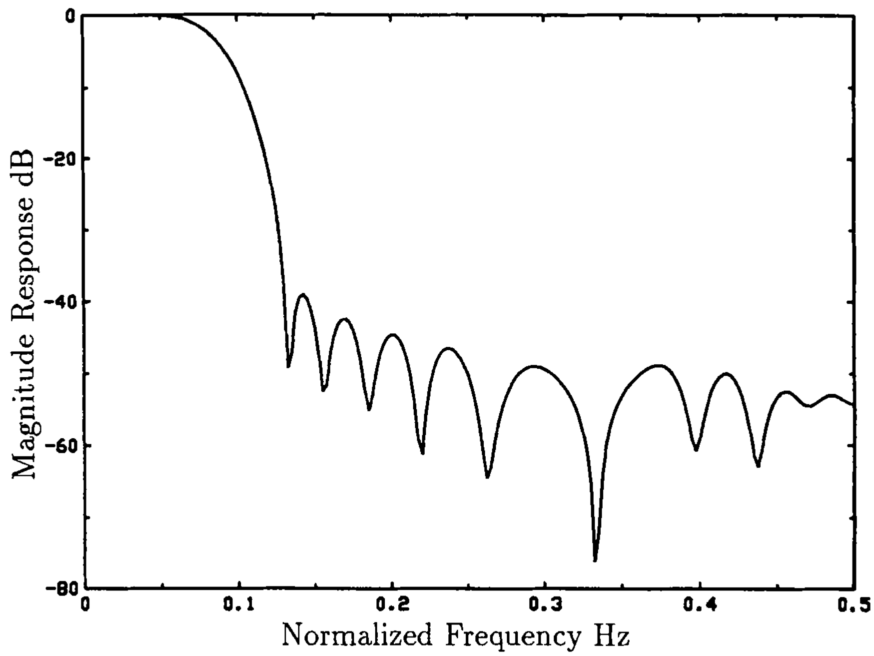


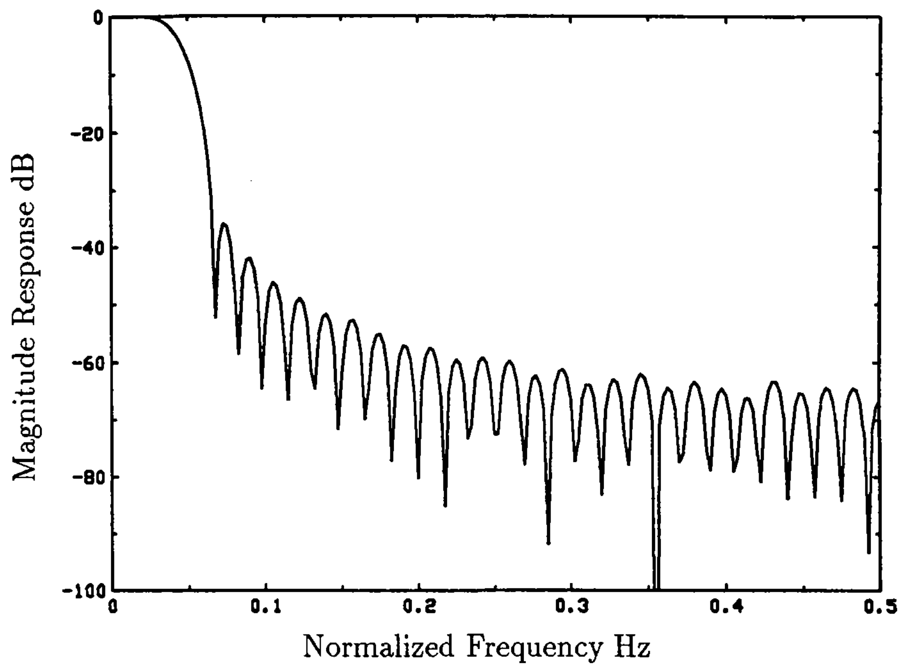
Fig. 5.2 Magnitude response of the lowpass filter for system T3. The weighting factors are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$.

formance (in terms of intersymbol interference and crosstalk distortions). Increasing γ_4 beyond 0.01 merely increases the number of iterations.

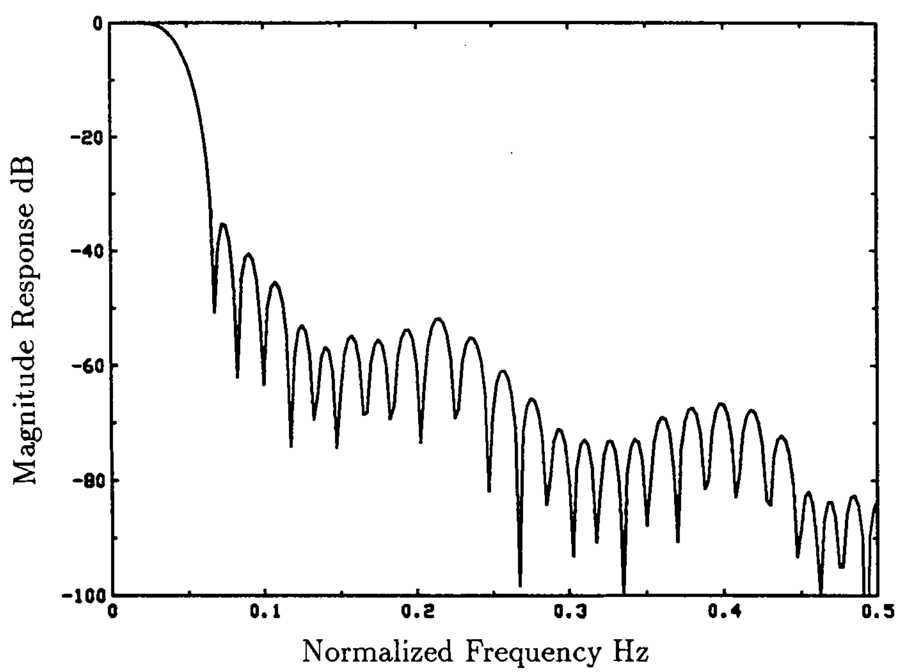
As an alternative to the Quasi-Newton procedure, the steepest descent algorithm was also attempted with the same initial conditions. At the beginning, there was a rapid decrease in the error. Then, there was a very slow decrease in the error but no signs of convergence even after many iterations.

5.4 Transmultiplexer Performance

The performance of the transmultiplexers is evaluated and compared for minimax filters and for filters designed by the method in this chapter. The transmultiplexers have six bands ($N = 6$) and use filters having an excess bandwidth of 52 percent



(a) The weighting factors are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 0, 0.01)$.



(b) The weighting factors are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 1, 0.01)$.

Fig. 5.3 Magnitude response of the lowpass filter for system T4.

($\beta = 0.52$). For systems T1, T2 and T3, the aim is to achieve a minimum stopband attenuation of about 40 dB. A stopband attenuation of about 35 dB is used for systems T4 and T5 since an excessively long prototype would be required for a 40 dB attenuation when using the minimax method.

For the one prototype systems (T1, T2, T4 and T5), a minimax linear phase $H(z)$ is designed by the McClellan-Parks algorithm [25] such that its frequency response approximates the square root of a raised cosine spectrum. The factorable minimax method is used for T3. The resulting prototypes $H(z)$ and $H(z^{-1})$ are not linear phase. For T1 and T2, the prototype has 77 taps. For T3, a 30 tap filter results. For T4 and T5, a 99 tap prototype is used. Equiripple designs are obtained by a weighting function equal to unity. Figure 4.2 (design Example 1 in Chapter 4) depicts the magnitude response of the equiripple Nyquist filter whose 30 tap minimum and maximum phase parts are used in T3 for the performance study.

We also design nonequiripple responses for the transmultiplexers. For the one prototype systems, the weighting function $W(\omega)$ is unity in the passband and the transition band. In the stopband, an increasing weight is used,

$$W(\omega) = \frac{200}{2\pi}(\omega - \omega_s) + 1 \qquad (5.10)$$

for $\omega_s \leq \omega \leq \pi$. In the case of T3, the factorable minimax method is based exclusively on stopband control and hence, allows for weighting only in the stopband. We use $W(\omega)$ as above for $\omega_s \leq \omega \leq \pi$. These filters, with a stopband attenuation increasing towards π, should achieve a higher crosstalk suppression. In all cases, the minimum stopband attenuation (at the stopband edge) is essentially the same for the equiripple and nonequiripple filters. However, the attenuation at the high frequencies for the nonequiripple designs is 58 dB (77 tap prototype for T1 and T2), 52 dB (30 tap filter for T3) and 54 dB (99 tap filter for T4 and T5).

Using the new method involving an unconstrained minimization of the error function E, we design a 33 tap filter for systems T1 and T2, a 30 tap filter for system T3 and a 59 tap filter for transmultiplexers T4 and T5. For systems T1, T2, T4 and T5, the initial condition for the optimization corresponds to an equiripple linear phase filter (with unity gain at zero frequency) having a frequency response that is a minimax approximation of the square root of a raised cosine spectrum. For system T3, the initial condition corresponds to an equiripple minimum phase filter (with unity gain at zero frequency) designed by the factorable minimax method. The weighting factors used are $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (100, 1, 0, 0.01)$ and $(100, 1, 1, 0.01)$. The design examples in the previous section correspond to those used here in the performance study. The minimum stopband attenuations (at the stopband edge) are approximately equal whether crosstalk is taken into account or not ($\gamma_3 = 1$ or $\gamma_3 = 0$).

In measuring the performance of the transmultiplexers, we consider the normalized peak distortion D_{P} and the normalized root mean-square (RMS) distortion D_{RMS} for the intersymbol interference. These performance measures have been used in Chapter 4 to compare the factorable minimax design method with the McClellan-Parks approach. For the lth terminal, $D_{\mathrm{P}}(l)$ is

$$D_{\mathrm{P}}(l) = \frac{\sum\limits_{\substack{n \\ n \neq 0}} |t_{ll}(n)|}{|t_{ll}(0)|} \tag{5.11}$$

and $D_{\mathrm{RMS}}(l)$ is

$$D_{\mathrm{RMS}}(l) = \sqrt{\frac{\sum\limits_{\substack{n \\ n \neq 0}} t_{ll}^2(n)}{t_{ll}^2(0)}} \,. \tag{5.12}$$

Note that the factor E_{isi} in the error function only considers the mean-square distortion. The quantity $D_{\mathrm{P}}(l)$ as well as $D_{\mathrm{RMS}}(l)$ will be the same for all terminals in T2, T4 and T5. There will be some variation among the terminals in T1 and T3.

The normalized crosstalk power at terminal l, $D_{\mathrm{CRP}}(l)$ is the performance measure for the crosstalk. It is expressed as

$$\begin{aligned} D_{\mathrm{CRP}}(l) &= \frac{P_{ctk}(l)}{\frac{P_{\mathrm{s}}}{N^2}\sum\limits_{n} t_{ll}^2(n)} \\ &= \frac{\sum\limits_{\substack{k=0 \\ k \neq l}}^{N-1} \sum\limits_{n} t_{kl}^2(n)}{\sum\limits_{n} t_{ll}^2(n)} \end{aligned} \tag{5.13}$$

The output signal at terminal l contains two components, one desired term resulting from the corresponding input and an undesired factor due to crosstalk. At terminal l, the power of the desired component is the input signal power P_{s} multiplied by $\frac{1}{N^2}\sum_n t_{ll}^2(n)$. Dividing the total crosstalk power by the power of the desired component establishes the normalized crosstalk power $D_{\mathrm{CRP}}(l)$ which can be thought of as a crosstalk to signal ratio.

Tables 5.2, 5.3 and 5.4 show the values of $D_{\mathrm{P}}(l)$, $D_{\mathrm{RMS}}(l)$ and $D_{\mathrm{CRP}}(l)$ (in dB) for the transmultiplexers when $N = 6$. Only the values for the first three output terminals are provided since symmetry gives the same results for the other three terminals†. We proceed to analyze the results and compare the two design methods.

† Note that for system T5 with $\gamma_3 = 1$, the optimization algorithm did not converge. A fixup involved using only the crosstalk terms having sidebands separated by no more than $\pi/2$.

Intersymbol Interference Suppression

In Section 5.1.1, we identified two potential sources of intersymbol interference. These are (1) the limitation of the design procedure in giving filters such that the Nyquist criterion is not exactly satisfied and (2) the fact that the prototypes are not bandlimited. These causes of intersymbol interference are reflected in Tables 5.2 and 5.3. In the forthcoming analysis, we refer to these sources of intersymbol interference as Source (1) and Source (2). Also, our observations are confined to the first three terminals of the transmultiplexers. However, these observations will hold for the corresponding last three terminals due to symmetry.

First, consider the minimax designs. Source (1) is the only potential cause of intersymbol interference in systems T2, T4 and T5 and at terminal 0 of T1 and T3. There is no intersymbol interference at terminal 0 of T3 since the factorable minimax method assures a Nyquist characteristic. For the other cases, a minimax design that approximates the square root of a Nyquist characteristic leads to intersymbol interference. Regarding terminals 1 and 2 of transmultiplexer T1, both Source (1) and Source (2) contribute to intersymbol interference. However, the small variation in the values of D_{P} and D_{RMS} for T1 shows that Source (2) is not severe. At terminals 1 and 2 of T3, only Source (2) contributes to intersymbol interference. The low normalized peak and RMS distortions for terminals 1 and 2 of T3 again show that Source (2) is not severe. In fact, T3 outperforms the other systems indicating that Source (1) is the dominant cause of intersymbol interference. Applying an increasing frequency weight in the stopband does not affect the normalized peak and RMS distortions significantly

Transmultiplexer	$D_P(l)$ in dB minimax design constant $W(\omega)$	$D_P(l)$ in dB minimax design increasing $W(\omega)$
T1	$-30 \; -29 \; -31$	$-29 \; -29 \; -30$
T2	$-30 \; -30 \; -30$	$-29 \; -29 \; -29$
T3	$-\infty \; -39 \; -39$	$-\infty \; -48 \; -48$
T4	$-26 \; -26 \; -26$	$-23 \; -23 \; -23$
T5	$-26 \; -26 \; -26$	$-23 \; -23 \; -23$

(a) Minimax designs

Transmultiplexer	$D_P(l)$ in dB optimized design $\gamma_3 = 0$	$D_P(l)$ in dB optimized design $\gamma_3 = 1$
T1	$-56 \; -55 \; -54$	$-56 \; -56 \; -56$
T2	$-56 \; -56 \; -56$	$-56 \; -56 \; -56$
T3	$-99 \; -49 \; -49$	$-92 \; -78 \; -82$
T4	$-56 \; -56 \; -56$	$-57 \; -57 \; -57$
T5	$-56 \; -56 \; -56$	$-56 \; -56 \; -56$

(b) Optimized designs with $(\gamma_1, \gamma_2, \gamma_4) = (100, 1, 0.01)$.

Table 5.2 Peak distortion (in dB) for transmultiplexers T1 to T5. Entries along a row refer to output terminals $l = 0$, 1 and 2 respectively.

except for terminals 1 and 2 of system T3. An enhanced stopband response (due to an increasing frequency weight) diminishes the effect of Source (2) and leads to lower normalized peak and RMS distortions at terminals 1 and 2 of system T3.

Now, consider the optimized design for the one prototype systems (T1, T2, T4 and T5). Source (1) leads to intersymbol interference in all the systems. Source (2) only affects terminals 1 and 2 of system T1. However, Source (1) is the dominant cause of intersymbol interference. This is exemplified by the fact that there is very little variation in the values of D_P and D_{RMS} for T1. The normalized peak and RMS distortions are not significantly different for the cases $\gamma_3 = 0$ and $\gamma_3 = 1$.

Transmultiplexer	$D_{RMS}(l)$ in dB minimax design constant $W(\omega)$	$D_{RMS}(l)$ in dB minimax design increasing $W(\omega)$
T1	−36 − 34 − 37	−34 − 34 − 34
T2	−36 − 36 − 36	−34 − 34 − 34
T3	−∞ − 45 − 45	−∞ − 54 − 54
T4	−31 − 31 − 31	−31 − 31 − 31
T5	−31 − 31 − 31	−31 − 31 − 31

(a) Minimax designs

Transmultiplexer	$D_{RMS}(l)$ in dB optimized design $\gamma_3 = 0$	$D_{RMS}(l)$ in dB optimized design $\gamma_3 = 1$
T1	−60 − 60 − 60	−60 − 60 − 60
T2	−60 − 60 − 60	−60 − 60 − 60
T3	−105 − 57 − 57	−96 − 83 − 88
T4	−62 − 62 − 62	−63 − 63 − 63
T5	−62 − 62 − 62	−62 − 62 − 62

(b) Optimized designs with $(\gamma_1, \gamma_2, \gamma_4) = (100, 1, 0.01)$.

Table 5.3 RMS distortion (in dB) for transmultiplexers T1 to T5. Entries along a row refer to output terminals $l = 0$, 1 and 2 respectively.

In the case of an optimized design for T3, the intersymbol interference at terminal 0 is only due to Source (1). However, both Source (1) and Source (2) affect terminals 1 and 2. In contrast to the one prototype systems, Source (2) is the major cause of intersymbol interference. This is revealed by the large difference in the normalized peak and RMS distortions for terminals 1 and 2 compared with terminal 0. The initial condition used in the optimization corresponds to a filter $H(z)$ that assures exact zero crossings in the impulse response of $H(z)H(z^{-1})$. The use of this initial condition results in an optimized filter $H(z)$ that sacrifices the zero crossing property of $H(z)H(z^{-1})$. However, the resulting intersymbol interference distortion is very low at terminal 0. A crosstalk weight ($\gamma_3 = 1$) leads to more distortion at terminal 0 and less distortion at terminals 1 and 2 compared to the case $\gamma_3 = 0$. For terminals 1 and 2 of T3, the bandlimitedness property is used to cancel terms in the input-output transfer function involving sidebands whose center frequencies are separated by $2\pi/3$. Source (2) contributes to intersymbol interference at these terminals. The enhanced stopband attenuation about $2\pi/3$ that results from the use of a positive crosstalk weight diminishes the effect of Source (2). This results in a lower intersymbol interference distortion at terminals 1 and 2.

Crosstalk Suppression

The QAM systems (T1, T2 and T3) generally achieve a much lower normalized crosstalk power than the VSB transmultiplexers (T4 and T5) primarily because QAM systems exhibit many more crosstalk functions that are exactly zero. An exception arises for the optimized design with $\gamma_3 = 0$. In this case, T4 and T5 achieve a lower normalized crosstalk power than T3. However, this occurs by using a filter in T4 and T5 that has more taps and a better overall stopband response than the filter used in T3. Also, we notice that the crosstalk power is exactly zero for terminal 2 of T1. Among the QAM systems, T1 and T2 outperform T3 but at the expense of more filter coefficients (the disparity in the number of coefficients is much more for the minimax designs). For a minimax design, an increasing frequency weight diminishes the crosstalk power as anticipated. For the optimized design, a positive crosstalk weight ($\gamma_3 = 1$) results in a substantially lower crosstalk power than for a zero crosstalk weight.

Transmultiplexer	$D_{\text{CRP}}(l)$ in dB minimax design constant $W(\omega)$	$D_{\text{CRP}}(l)$ in dB minimax design increasing $W(\omega)$
T1	$-47\ \ -47\ \ -\infty$	$-65\ \ -65\ \ -\infty$
T2	$-47\ \ -47\ \ -47$	$-65\ \ -65\ \ -65$
T3	$-39\ \ -40\ \ -41$	$-47\ \ -49\ \ -48$
T4	$-25\ \ -25\ \ -25$	$-40\ \ -40\ \ -40$
T5	$-26\ \ -26\ \ -26$	$-43\ \ -44\ \ -41$

(a) Minimax designs

Transmultiplexer	$D_{\text{CRP}}(l)$ in dB optimized design $\gamma_3 = 0$	$D_{\text{CRP}}(l)$ in dB optimized design $\gamma_3 = 1$
T1	$-70\ \ -70\ \ -\infty$	$-87\ \ -87\ \ -\infty$
T2	$-70\ \ -70\ \ -70$	$-87\ \ -87\ \ -87$
T3	$-46\ \ -48\ \ -45$	$-74\ \ -77\ \ -73$
T4	$-54\ \ -54\ \ -54$	$-65\ \ -65\ \ -65$
T5	$-49\ \ -50\ \ -52$	$-60\ \ -60\ \ -61$

(b) Optimized designs with $(\gamma_1, \gamma_2, \gamma_4) = (100, 1, 0.01)$.

Table 5.4 Normalized crosstalk power (in dB) for transmultiplexers T1 to T5. Entries along a row refer to output terminals $l = 0$, 1 and 2 respectively.

Comparison of Minimax and Optimized Designs

The new optimized design approach is highly beneficial for the one prototype systems (T1, T2, T4 and T5). A much lower intersymbol interference and crosstalk distortion is achieved (even with a crosstalk weight of zero) with many fewer filter taps as compared to a minimax design. In addition, the optimized design allows for the flexibility of taking crosstalk into account by setting $\gamma_3 > 0$.

For system T3, we have proposed new minimax and optimized design approaches. For the performance study, the number of filter coefficients for the minimax and optimized designs are the same. Moreover, the minimax filters serve as initial conditions for the optimized design. The main advantage of the optimized design over the minimax design primarily lies in using a positive crosstalk weight to substantially diminish the crosstalk power. The optimized filters designed with a positive crosstalk weight lead to a lower crosstalk distortion (at all terminals) and a lower intersymbol interference distortion (at terminals 1 and 2) as compared to minimax filters. Without a crosstalk weight, there is no clear advantage of the optimized design. In fact, the factorable minimax approach with an increasing stopband weight and the optimized design with $\gamma_3 = 0$ lead to a similar performance. Finally, in contrast to the minimax approach, an optimized design will not give an $H(z)$ such that $H(z)H(z^{-1})$ is a Nyquist filter with exact zero crossings thereby resulting in residual intersymbol interference at terminal 0.

5.5 Design for the Complementary Subband Systems

Given the design method for the transmultiplexers, we now attempt to see whether this filter design approach carries over to the complementary subband systems. Note that the minimax design approaches can be used for both the transmultiplexers and their subband complements. The complementary subband systems have an input-output relationship $\hat{X}(z) = \frac{1}{N}T(z^N)X(z)$ if the prototypes are bandlimited where $T(z^N)$ is defined in Eqs. (3.15), (3.17), (3.19) and (3.21). In addition, perfect reconstruction is accomplished by satisfying the Nyquist criterion. With practical prototypes, there is residual aliasing in that the input-output relationship becomes $\hat{X}(z) = \frac{1}{N}T(z^N)X(z)$ + terms due to aliasing. In a practical design, the stopband edge frequency is restricted as in the case of transmultiplexers. In formulating a suitable error function, the factors E_{sb}, E_{isi} and the factor that avoids a zero solution ($(\mathbf{b}^T\mathbf{b} - 1)^2$ or $(\mathbf{h}^T\mathbf{h} - 1)^2$) are the same as for the transmultiplexers. The remaining question is about how to take aliasing into account. In general, the output of a subband system is a combination of a filtered input and filtered frequency shifted versions of the input. Even for a zero-mean white input, the filtered input is correlated with the filtered frequency shifted versions of the input. This makes it difficult to express the total power at the output due to aliasing in relation to the power of the desired component due to the input especially for an arbitrary N. However, filters can be designed by minimizing the error function having the factors E_{sb}, E_{isi} and the factor that avoids a zero solution. The filters that were previously designed with $\gamma_3 = 0$ can be used in the complementary subband systems†.

Similar error functions for designing a prototype for subband systems have been proposed in [19][43]. A subband system with two bands which accomplishes a natural

† Note that filters designed with $\gamma_3 = 1$ do not seem to perform any better (or any worse) with respect to suppression of aliasing than filters designed with $\gamma_3 = 0$.

cancellation of aliasing is the focus of [19][43]. The error functions are weighted linear combinations of two components. The first component is the stopband energy which in [19] is expressed as an integral and which in [43] is approximated as a sum over a dense grid. The second component is the mean-square distortion at the output. The actual expressions in [19] and [43] differ in that a time domain approach is used in the former and a frequency domain approach is used in the latter. The error function for our subband systems consisting of a weighted linear combination of the terms E_{sb}, E_{isi} and the term that avoids a zero solution is based on a time domain approach as in [19].

Chapter 6 Channel Distortion and Compensation

Until now, the investigation on modulated filter banks assumed that there is no channel distortion. However, a channel is present when data is transmitted from one location to another. This brings up the question of how to achieve reconstruction of the input data signals when there is channel distortion given that reconstruction can be accomplished in the absence of a channel. This chapter provides preliminary results that deal with this issue. Methods to configure a channel compensation filter to combat channel distortion are derived. Also, the performance of these methods is evaluated for a specific channel.

6.1 Combating Channel Effects

In a transmultiplexer, the composite signal passes through a single channel. The overall system is shown in Fig. 6.1 where the channel has a system function $Q(z)$. In attempting to alleviate the effects of the channel $Q(z)$, we assume that the combining and separation banks are configured to satisfy $\mathbf{A}(z)\mathbf{B}^T(z) = T(z^N)\mathbf{I}$. Therefore, in the absence of a channel, the transmultiplexer is crosstalk-free and has the same input-output transfer function for every pair of terminals†. This can be assured by the methods given in Chapter 2 and by the new modulated filter banks developed in this thesis. The problem is to specify a channel compensation filter that acts on the received composite signal and nullifies the channel distortion. The only compensation filter that accomplishes this is $1/Q(z)$ which is unstable if $Q(z)$ has zeros outside the unit circle. Approaches are formulated to configure a stable compensation filter that reinstates the crosstalk cancellation property and suppresses the resulting intersymbol interference. Then, simulations are done to compare the various approaches.

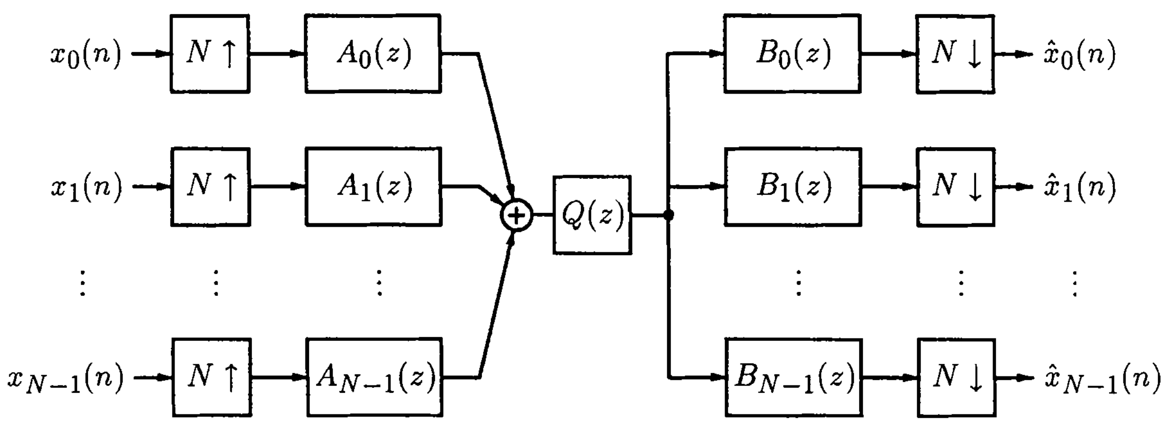


Fig. 6.1 Channel distortion in a transmultiplexer

6.1.1 Theoretical Development

In Chapter 2, the input-output relations for a transmultiplexer were given assuming no channel distortion. When a channel is present, the outputs $\hat{X}_i(z)$ (as in Fig. 6.1) are given by

$$\hat{\mathbf{X}}^T(z^N) = \frac{1}{N}\mathbf{X}^T(z^N)\mathbf{A}(z)\mathbf{Q}(z)\mathbf{B}^T(z) \tag{6.1}$$

† Note that this assumption includes the special case of perfect reconstruction.

where

$$\mathbf{Q}(z) = \text{Diag}\ [Q(z),\ Q(zW^{-1}),\ \cdots,\ Q(zW^{-N+1})]\ , \tag{6.2}$$

$$\mathbf{X}(z^N) = \begin{bmatrix} X_0(z^N) \\ X_1(z^N) \\ \vdots \\ X_{N-1}(z^N) \end{bmatrix}\ ,\quad \hat{\mathbf{X}}(z^N) = \begin{bmatrix} \hat{X}_0(z^N) \\ \hat{X}_1(z^N) \\ \vdots \\ \hat{X}_{N-1}(z^N) \end{bmatrix} \tag{6.3}$$

and $\mathbf{A}(z)$ and $\mathbf{B}(z)$ are defined as in Chapter 2. Since the system with no channel distortion ($Q(z) = 1$) eliminates crosstalk and has the same input-output transfer function for every pair of terminals, $\mathbf{A}(z)\mathbf{B}^T(z) = T(z^N)\mathbf{I}$. To cancel crosstalk with the presence of a channel, one needs to satisfy the augmented equation

$$\mathbf{A}(z)\mathbf{Q}(z)\mathbf{B}^T(z) = S(z^N)T(z^N)\mathbf{I}\ . \tag{6.4}$$

In the sequel, it is assumed that $Q(z)$ is a stable function. No restriction on the zeros of $Q(z)$ is imposed. A channel compensation filter $E(z)$ that acts on the received composite signal is equivalent to modifying the separation filters to $B'_k(z) = B_k(z)E(z)$ for $k = 0$ to $N-1$. Then, a new separation filter matrix $\mathbf{B}'(z) = \mathbf{B}(z)\mathbf{R}(z)$ results where

$$\mathbf{R}(z) = \text{Diag}\ [E(z),\ E(zW^{-1}),\ \cdots,\ E(zW^{-N+1})]\ . \tag{6.5}$$

If $\mathbf{R}(z)$ is chosen such that $\mathbf{Q}(z)\mathbf{R}(z) = S(z^N)\mathbf{I}$, Eq. (6.4) becomes

$$\begin{aligned} \mathbf{A}(z)\mathbf{Q}(z)[\mathbf{B}'(z)]^T &= \mathbf{A}(z)\mathbf{Q}(z)\mathbf{R}(z)\mathbf{B}^T(z) \\ &= S(z^N)\mathbf{I}\mathbf{A}(z)\mathbf{B}^T(z) \\ &= S(z^N)T(z^N)\mathbf{I}\ . \end{aligned} \tag{6.6}$$

In choosing $\mathbf{R}(z)$, the stability of $E(z)$ must be ensured.

The special case in which the channel response $Q(z)$ is itself a function of z^N ensures that $\mathbf{A}(z)\mathbf{Q}(z)\mathbf{B}^T(z)$ remains a function of z^N and consequently, no crosstalk is introduced by the channel [44]. A special case is when $Q(z)$ is a pure delay of the form z^{-mN}. This is equivalent to applying delay factors to the combining and separation filter banks. Then, the perfect reconstruction property is preserved for an identity transmultiplexer as discussed in Chapter 2.

An obvious solution to $\mathbf{Q}(z)\mathbf{R}(z) = S(z^N)\mathbf{I}$ is to choose $\mathbf{R}(z) = \mathbf{Q}^{-1}(z)$. This makes $S(z^N) = 1$ and $E(z) = 1/Q(z)$. However, this solution is inappropriate if $Q(z)$ has zeros outside the unit circle since an unstable compensation filter $E(z)$ results.

To achieve crosstalk cancellation, $\mathbf{R}(z)$ is set to be

$$\mathbf{R}(z) = \Phi(z^N)\mathbf{I}\mathbf{C}_Q(z) \tag{6.7}$$

where $\Phi(z^N)$ is any arbitrary function of z^N and $\mathbf{C}_Q(z)$ is the cofactor matrix of $Q(z)$ given by

$$\mathbf{C}_Q(z) = \begin{bmatrix} \prod_{\substack{i=0\\ i\neq 0}}^{N-1} Q(zW^{-i}) & 0 & \cdots & 0 \\ 0 & \prod_{\substack{i=0\\ i\neq 1}}^{N-1} Q(zW^{-i}) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \prod_{\substack{i=0\\ i\neq N-1}}^{N-1} Q(zW^{-i}) \end{bmatrix} . \tag{6.8}$$

Then,

$$\begin{aligned} \mathbf{Q}(z)\mathbf{R}(z) &= \Phi(z^N) \prod_{i=0}^{N-1} Q(zW^{-i})\mathbf{I} \\ &= S(z^N)\mathbf{I} \end{aligned} \tag{6.9}$$

and

$$E(z) = \Phi(z^N) \prod_{i=1}^{N-1} Q(zW^{-i}) . \tag{6.10}$$

The channel $Q(z)$ and the compensation filter $E(z)$ introduce an extra term $S(z^N)$ in the input-output transfer function. The overall input-output transfer function is $S(z^N)T(z^N)$. The compensation filter $E(z)$ can be thought of as being composed of two filters. The filter with system function $\prod_{i=1}^{N-1} Q(zW^{-i})$ when cascaded with $Q(z)$ can be viewed as a composite channel $\prod_{i=0}^{N-1} Q(zW^{-i})$, which being a function of z^N, ensures the cancellation of crosstalk. However, residual intersymbol interference remains. The other filter $\Phi(z^N)$ should be a function of z^N to preserve the crosstalk cancellation property. However, its actual role is to suppress the residual intersymbol interference admitted by the factor $S(z^N)$. We ignore the intersymbol interference admitted by the factor $T(z^N)$ since it can be suppressed or even cancelled by designing the filter banks. In effect, the channel compensation filter consists of two components, one which exactly cancels crosstalk and one which suppresses intersymbol interference. Since $Q(z)$ is stable, it follows that $E(z)$ is stable providing $\Phi(z^N)$ is stable. Based on the specification for $\mathbf{R}(z)$, different approaches of choosing $\Phi(z^N)$ are given.

6.1.1.1 Choices for $\Phi(z^N)$

Method 1

The simplest method, namely, $\Phi(z^N) = 1$ does not attempt to control the intersymbol interference. It introduces the factor $S(z^N) = \prod_{i=0}^{N-1} Q(zW^{-i})$ in the overall input-output transfer function.

Method 2

A second procedure alleviates the problem of a high order input-output transfer function that is present in the previous approach. Suppose $Q(z)$ is split up as

$$Q(z) = \frac{N(z)}{D(z)} = \frac{U_+(z)U_-(z)}{D(z)} , \tag{6.11}$$

where $U_+(z)$ contains the zeros of $Q(z)$ within the unit circle and $U_-(z)$ contains the zeros of $Q(z)$ on and outside the unit circle. Since $Q(z)$ is assumed to be stable, $D(z)$ has all its zeros within the unit circle. By setting

$$\Phi(z^N) = \prod_{i=0}^{N-1} \frac{D(zW^{-i})}{U_+(zW^{-i})} , \tag{6.12}$$

we get a lower order factor in the input-output transfer function

$$S(z^N) = \prod_{i=0}^{N-1} U_-(zW^{-i}) , \tag{6.13}$$

and a stable channel compensation filter

$$E(z) = \frac{D(z)}{U_+(z)} \prod_{i=1}^{N-1} U_-(zW^{-i}) . \tag{6.14}$$

Since the order of the overall input-output transfer function is reduced over that of Method 1, the resulting time span of the intersymbol interference is shortened.

Method 3

Assume that the original transfer function $T(z^N)$ is allpass and that $Q(z)$ has no zeros on the unit circle (in analogy with the development in [18]). Now, we proceed to examine whether the allpass property of the input-output transfer function can be preserved. Setting

$$\Phi(z^N) = \prod_{i=0}^{N-1} \frac{D(zW^{-i})}{U_+(zW^{-i})U_-(z^{-1}W^{i})} \tag{6.15}$$

renders a new allpass factor

$$S(z^N) = \prod_{i=0}^{N-1} \frac{U_-(zW^{-i})}{U_-(z^{-1}W^i)} \, . \tag{6.16}$$

and a stable channel compensation filter

$$E(z) = \frac{D(z)}{U_+(z)} \frac{\prod_{i=1}^{N-1} U_-(zW^{-i})}{\prod_{i=0}^{N-1} U_-(z^{-1}W^i)} \, . \tag{6.17}$$

Method 3 preserves the allpass property of the input-output transfer function but introduces an infinite time span for the intersymbol interference.

Method 4

So far, we have presented methods that either control the time span or the allpass nature of the input-output transfer function. Now, we attempt to choose an FIR $\Phi(z^N)$ so as to suppress the intersymbol interference. Given that

$$\begin{aligned} S(z^N) &= \Phi(z^N) \prod_{i=0}^{N-1} Q(zW^{-i}) \\ &= \Phi(z^N)\Psi(z^N) \end{aligned} \tag{6.18}$$

or equivalently $S(z) = \Phi(z)\Psi(z)$, we determine the coefficients of an FIR $\Phi(z)$ to minimize the mean-square intersymbol interference $\sum_{n\neq 0} s^2(n)$. Since $s(n) = \phi(n) \star \psi(n)$, it can be shown that $\sum_{n\neq 0} s^2(n) = \boldsymbol{\phi}^{\mathrm{T}} \Psi \boldsymbol{\phi}$ where $\boldsymbol{\phi}$ is the column vector of coefficients of $\Phi(z)$ and Ψ is a positive definite symmetric matrix whose entries $\Psi(k,l)$ are given by

$$\Psi(k,l) = \sum_{\substack{n \\ n\neq 0}} \psi(n-k)\psi(n-l) \, . \tag{6.19}$$

To avoid the trivial solution $\Phi(z) = 0$, we impose the constraint $\boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi} = 1$. Then, $\boldsymbol{\phi}$ is the eigenvector corresponding to the minimum eigenvalue of Ψ. Note that Method 4 can be viewed as attempting to approximate the inverse of the composite channel $\prod_{i=0}^{N-1} Q(zW^{-i})$.

Method 5

An alternative method to suppress the intersymbol interference is to choose $\Phi(z^N)$ to be

$$\Phi(z^N) = \Delta(z^N) \prod_{i=0}^{N-1} \frac{D(zW^{-i})}{U_+(zW^{-i})} . \tag{6.20}$$

Then,

$$S(z^N) = \Delta(z^N) \prod_{i=0}^{N-1} U_-(zW^{-i}) . \tag{6.21}$$

An FIR $\Delta(z^N)$ is determined to suppress the mean-square intersymbol interference. As compared to Method 4, Method 5 only performs an approximation of the inverse of a maximum phase function that contains the zeros of $Q(z)$ on and outside the unit circle. A factor of $\Phi(z^N)$ exactly cancels the zeros and poles of $Q(z)$ within the unit circle.

Summary of Methods

Table 6.1 shows the compensation filter $E(z)$ and the overall input-output transfer functions $T(z^N)S(z^N)$ resulting from the methods presented above. Suppose we have an FIR channel $Q(z)$. This leads to either an FIR or IIR compensation filter depending on the method utilized. Assuming that $T(z^N)$ is an FIR function (this is often the case since perfect reconstruction is desired), the overall input-output transfer function is FIR in Methods 1, 2, 4 and 5. Method 3 is only useful for an allpass $T(z^N)$ and renders an IIR input-output transfer function.

Method	Compensation Filter Function	Input-Output Transfer Function
1	$\prod_{i=1}^{N-1} Q(zW^{-i})$	$T(z^N)\prod_{i=0}^{N-1} Q(zW^{-i})$
2	$\frac{D(z)}{U_+(z)}\prod_{i=1}^{N-1} U_-(zW^{-i})$	$T(z^N)\prod_{i=0}^{N-1} U_-(zW^{-i})$
3	$\frac{D(z)}{U_+(z)}\frac{\prod_{i=1}^{N-1} U_-(zW^{-i})}{\prod_{i=0}^{N-1} U_-(z^{-1}W^{i})}$	$T(z^N)\prod_{i=0}^{N-1}\frac{U_-(zW^{-i})}{U_-(z^{-1}W^{i})}$
4	$\Phi(z^N)\prod_{i=1}^{N-1} Q(zW^{-i})$	$T(z^N)\Phi(z^N)\prod_{i=0}^{N-1} Q(zW^{-i})$
5	$\Delta(z^N)\frac{D(z)}{U_+(z)}\prod_{i=1}^{N-1} U_-(zW^{-i})$	$T(z^N)\Delta(z^N)\prod_{i=0}^{N-1} U_-(zW^{-i})$

Table 6.1 Channel compensation filter and overall input-output transfer functions for the methods

An IIR channel results in IIR compensation filters for all of the methods. However, Methods 1, 3 and 4 produce an IIR input-output transfer function while Methods 2 and 5 still produce an FIR input-output transfer function (under the assumption that $T(z^N)$ is FIR). Methods 2, 3 and 5 involve additional computation to split the numerator of $Q(z)$ into its minimum and maximum phase parts.

For the special case when $Q(z)$ is a function of z^N, crosstalk is not introduced.

Then, the compensation filters for Methods 1 and 4 assume a special form. Method 1 renders a compensation filter $E(z) = (N-1)Q(z)$ which is not particularly appropriate since crosstalk is already absent and no specific control of the intersymbol interference is provided for. In Method 4, the form of the compensation filter should reduce to $E(z) = \Phi(z^N)$ as no additional factor is necessary to cancel crosstalk. Then, $\Phi(z^N)$ will approximate the inverse of $Q(z)$. Note that for a general $Q(z)$ (not a function of z^N), using a compensation filter to suppress the mean-square intersymbol interference does not result in crosstalk cancellation.

6.1.2 Performance Evaluation

We evaluate the performance of a transmultiplexer when the different channel compensation filters are used. Consider the two band ($N = 2$) version of transmultiplexer T3 with $G(z) = H(z^{-1})$. The filters $H(z)$ and $H(z^{-1})$ are obtained by the factorable minimax approach such that $T(z^N) = N$ (an identity transmultiplexer). Therefore, with practical filters, both intersymbol interference and crosstalk are exactly cancelled when no channel distortion is present. The presence of a channel and a compensation filter reinstates the exact crosstalk cancellation property and introduces the extra term $S(z^N)$ in the input-output transfer function. The residual intersymbol interference is only due to $Q(z)$ and $E(z)$ and not the practical filters used in the combining and separation filter banks. Therefore, the evaluation of the performance only depends on the compensation filter. By calculating $s(n)$ (the inverse z-transform of $S(z)$), we measure both the normalized peak distortion D_P and the normalized RMS distortion D_{RMS}. The normalized peak distortion is given by

$$D_{\mathrm{P}} = \frac{\sum\limits_{\substack{n \\ n \neq 0}} |s(n)|}{|s(0)|} . \tag{6.22}$$

The normalized RMS distortion is given by

$$D_{\mathrm{RMS}} = \sqrt{\frac{\sum\limits_{\substack{n \\ n \neq 0}} s^2(n)}{s^2(0)}} . \tag{6.23}$$

In many communications applications, the multiplexed output of the combining bank is converted to a lowpass analog signal, modulated, sent across a channel and demodulated back to baseband. Then, continuous time to discrete time (C/D) conversion takes place prior to the action of the separation bank. First, forming a lowpass analog signal from the discrete time output of the combining bank involves converting the discrete time signal into an impulse train and passing the impulse train through a lowpass analog filter (D/C conversion). Note that C/D conversion is equivalent to sampling the continuous time input. This overall process is equivalent to transmitting the lowpass analog signal (formed by D/C conversion) over a lowpass equivalent channel and then performing C/D conversion as shown in Fig. 6.2. The D/C and C/D conversions are performed in phase† and at the same sampling rate $f_s = 1/T_s$.

For our performance study, we need a discrete time equivalent $Q(z)$ that models the system of Fig. 6.2. The process of D/C conversion translates the discrete time input into an impulse train and uses an ideal raised cosine filter with 50 percent roll-off to get the lowpass analog signal. In the absence of a channel, the discrete time

† Note that if the D/C and C/D conversions are done out of phase, this can be modelled as an extra linear phase component in the channel.

equivalent $Q(z) = 1$. We consider a lowpass equivalent channel with a cubic phase characteristic (parabolic group delay) $\theta(\Omega)$ given by [45]

$$\theta(\Omega) = -\frac{\rho}{3\pi^2}(\Omega T_s)^3 \; . \tag{6.24}$$

In effect, we are using a channel with a heavily distorted phase response that becomes more severe with increasing ρ. Such a phase nonlinearity exists over telephone channels and has been used to study the performance of multicarrier modems [23].

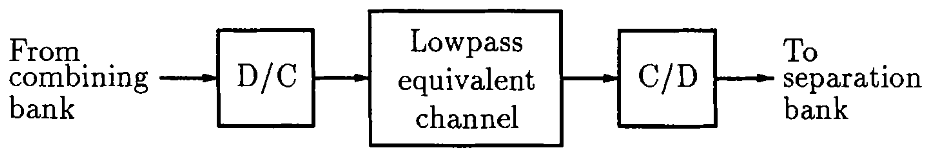


Fig. 6.2 Transmission over an analog channel

Specifically, we consider the discrete time equivalent response $q(n)$ for the case $\rho = 5$. This is representative of the group delay distortion that is seen by a high speed modem over a telephone channel. The discrete time response $q(n)$ diminishes rapidly with $|n|$. An FIR $Q(z)$ with 43 coefficients spans the significant part of the response. The magnitude response of $Q(z)$ is flat up to the quarter sampling frequency and then decreases by 6 dB at the half sampling frequency due to aliasing effects. The group delay is parabolic up to the quarter sampling frequency and then becomes more severe.

In calculating the normalized peak and RMS distortions for the first three methods, the reference coefficient that leads to the minimum distortion is aligned with the zeroth time index. This is equivalent to applying an additional time advance to the compensation filter. Although the impulse response is infinite in extent for Method

3, lower bounds for the normalized peak and RMS distortions are computed by considering the first 60 samples. For Methods 4 and 5, the eigenvector corresponding to the minimum eigenvalue of the positive definite matrix is of dimension 61. Therefore, the component of the compensation filter involving the FIR least-squares approximation (denoted by $\Phi(z^N)$ or $\Delta(z^N)$) has 61 nonzero coefficients. In addition, the coefficients of $\Phi(z^N)$ or $\Delta(z^N)$ are centered about the zeroth time index. This time index corresponds to the best reference coefficient of $s(n)$ without the least-squares filter.

The normalized peak and RMS distortions resulting from Methods 1, 2 and 3 are much larger than for Methods 4 and 5 primarily because there is no explicit suppression of the intersymbol interference. Specifically, Methods 1, 2 and 3 give peak distortions of 1.47, 0.46 and 1.11 respectively and RMS distortions of 0.91, 0.37 and 0.58 respectively. Of the first three approaches, Method 2 achieves the lowest distortion and constrains the time span of the intersymbol interference. Method 3 is highly specific to preserving a stable IIR allpass transfer function. Even though the impulse response dies out with time, a large distortion results. Methods 4 and 5 are successful in that they result in very low peak and RMS distortions, all of which are below 10^{-4}.

6.1.3 Application to Specific Systems

The methods used to configure a channel compensation filter assume that the transmultiplexer is crosstalk-free and results in the same input-output transfer function for every pair of terminals in the absence of a channel. The derivation is general in that there are no specific requirements on the form of the combining and separation filter banks. In addition, N can be any integer. Therefore, the channel compensation filters are applicable to two band QMF banks, the two band Smith-Barnwell structure, the N band systems configured by the use of a matrix formalism and N band lossless filter banks (see Chapter 2).

With bandlimited prototypes, transmultiplexers T1 to T5 satisfy the assumptions for configuring the channel compensation filters. Therefore, the channel compensation methods of this chapter can be applied to these transmultiplexers. Note that with practical filters, residual crosstalk is admitted. Suppose the channel compensation schemes are applied to the transmultiplexers that use practical filters. Then, the extra factor $S(z^N)$ is introduced in the input-output transfer function and the crosstalk functions. Specifically, the crosstalk functions become $T_{kl}(z^N)S(z^N)$ where $T_{kl}(z^N)$ are the crosstalk functions of the system in the absence of a channel. Methods 4 and 5 are particularly effective in that the factor $S(z^N)$ is approximately a constant. Then, the normalized crosstalk power will be about the same as the crosstalk power that is admitted in the absence of a channel.

6.1.4 Channel Effects in a Subband System

Channel distortion is introduced in a subband system when each of the intermediate signals formed after sampling rate compression is passed through a channel. Given that the original system with no channel distortion eliminates aliasing, the procedure given in [18] describes how to modify the synthesis filters to combat channel distortion. Just as in our approaches for a transmultiplexer, no specific assumptions about the filter banks or the number of bands N are made. Each of the synthesis filters is modified by a different factor that depends on the system function of each of the channels such that the cancellation of aliasing is reinstated. Our methods modify the separation filters by the same factor†. Our Methods 1, 2 and 3 are analogous to the approaches in [18]. In addition, we have proposed two additional procedures to control the intersymbol interference. The modification approaches for both subband systems and transmultiplexers do not assure perfect reconstruction. Since the subband systems S1 to S5 are alias-free with bandlimited filters, the compensation schemes in [18] apply.

† We can extend our approach to allow for different compensation filters $E_k(z)$ in each band. If $Q(z)E_k(z)$ is a function of z^N for each k, crosstalk is cancelled. Different input-output transfer functions will result for each pair of terminals.

Chapter 7 Conclusions

The thesis has explored a class of transmultiplexers that use modulated filter banks. Modulated filter banks use bandpass versions of a lowpass prototype. We have also dealt with subband systems which are complementary to transmultiplexers. In this chapter, a list of the contributions that have arisen from the investigation are presented. Then, a summary of the entire thesis is given. Recommendations for future research are outlined.

7.1 Contributions

The contributions resulting from the research are as follows:

1. A synthesis procedure based on a bandlimited lowpass prototype was developed for transmultiplexers that use modulated filter banks. The aim is to cancel crosstalk and maintain the same input-output transfer function for every pair of terminals.

2. As a consequence of the synthesis procedure, five bandwidth efficient transmultiplexers emerge.

3. The systems can be interpreted from a communications point of view. Three of the systems implement multicarrier Quadrature Amplitude Modulation (QAM). The other two implement multicarrier Vestigial Sideband Modulation (VSB).

4. The two band case was examined in more detail. This led to the synthesis of new two band transmultiplexers.

5. The synthesized transmultiplexers can be converted into new subband systems.

6. New design methods for a practical FIR lowpass prototype were formulated to suppress intersymbol interference and crosstalk.

   (a) The minimax designs take intersymbol interference into account.

   (b) The designs based on the minimization of an error function attempt to suppress both intersymbol interference and crosstalk. In fact, this design is based on an analysis of the transmultiplexers with respect to both intersymbol interference and crosstalk for practical filters.

7. The performance of the transmultiplexers with the designed practical filters was evaluated. This performance evaluation allows us to compare the transmultiplexers and the two design methods.

8. Filter design methods for the subband complements were described based on the approaches for transmultiplexers.

9. In the presence of channel distortion in transmultiplexers, five approaches to configure channel compensation filters were formulated to cancel crosstalk. The performance of these methods was compared with respect to the suppression of the resulting intersymbol interference given a realistic channel.

7.2 Summary

The motivation behind the investigation was to develop alternate configurations for transmultiplexers that use modulated filter banks. This was accomplished by formulating a synthesis procedure based on a bandlimited lowpass prototype (stopband response is exactly zero). The synthesis procedure is a constructive approach for developing new systems. As a result, five transmultiplexers (T1 to T5) were configured such that: (1) The input-output transfer function between each pair of terminals is the same and (2) The crosstalk is cancelled. Four of the systems are new while T4 resembles an existing modulated filter bank. Transmultiplexers T1, T2, T4 and T5 are based on one prototype $H(z)$. System T3 uses two prototypes $H(z)$ and $G(z)$.

Each of the transmultiplexers implements a form of Frequency Division Multiplexing (FDM) without the use of guard bands. Therefore, the full channel bandwidth is utilized by allowing for spectral overlap among the filters. In addition, the transmultiplexers are bandwidth efficient. Consider the case in which each input signal to the transmultiplexer is sampled at f_0 Hz. Then, the total information rate is $f_s = Nf_0$ samples/second where f_s is the sampling rate of the composite signal which occupies a bandwidth of $f_s/2$ Hz. The bandwidth efficiency of each of the systems is the ratio of the information rate (f_s samples/second) to the total bandwidth ($f_s/2$ Hz) and is equal to 2 samples/second/Hz. The synthesized transmultiplexers are bandwidth efficient in that the full information in each input is transmitted and the inputs are recovered.

Although all of the transmultiplexers accomplish FDM, a further interpretation from a communications point of view can be made. Three of the new systems (T1, T2 and T3) implement multicarrier Quadrature Amplitude Modulation (QAM). Two signals are sent in quadrature at each repeating center frequency. The other two (T4 and T5) accomplish multicarrier Vestigial Sideband Modulation (VSB) in which one signal is sent at each distinct frequency.

The N band transmultiplexers T1 to T5 can be converted into subband complements S1 to S5 respectively. Systems S1, S2, S3 and S5 are new while S4 resembles an existing system. Subband systems S1, S2 and S3 allow for repeated center frequencies. System S1 is an N band generalization of the two band QMF bank. System S3 is an N band generalization of a two band, two prototype system. For a particular case ($G(z) = H(z^{-1})$), S3 is lossless and is an N band generalization of the two band Smith-Barnwell structure. Both S4 and S5 use distinct center frequencies.

Transmultiplexers T1 through T5 have each been configured with bandlimited filters such that (1) the input-output transfer function is the same for every pair of corresponding terminals and (2) crosstalk is cancelled. In addition, satisfying the Nyquist criterion eliminates intersymbol interference and hence, achieves perfect reconstruction. For the one prototype systems (T1, T2, T4 and T5) specified by a lowpass $H(z)$, $H^2(z)$ should be a Nyquist filter. For transmultiplexer T3 specified by two lowpass filters $H(z)$ and $G(z)$, $H(z)G(z)$ should be a Nyquist filter. Since bandlimited filters cannot be designed and the Nyquist criterion may not be exactly satisfied, neither intersymbol interference nor crosstalk is exactly cancelled. New design methods for a practical FIR lowpass prototype were introduced with the added aim of suppressing intersymbol interference and crosstalk.

One of the design methods is based on a minimax approach to achieve a lowpass response. In addition, the desired Nyquist characteristic is taken into account. For the one prototype systems, there is an inherent difficulty in designing a lowpass $H(z)$ such that $H^2(z)$ exactly satisfies the Nyquist criterion. We used the McClellan-Parks algorithm to get a linear phase lowpass $H(z)$ that approximates the square root of a raised cosine response. For system T3 with $G(z) = H(z^{-1})$, a lowpass $H(z)$ can be designed such that $H(z)H(z^{-1})$ is a Nyquist filter. This is the advantage of using two prototypes in configuring T3.

Two new approaches called factorable minimax methods were formulated to design a lowpass Nyquist filter $H(z)H(z^{-1})$ having a Chebyshev stopband response. Both methods are iterative and four iterations were found to be sufficient in our examples to resolve the coefficients. The main advantages of the design techniques are that the polynomial factorization complexity in finding the minimum phase part $H(z)$ is considerably eased and that arbitrary frequency weighting can be applied without additional computational overhead. Although the two design approaches should theoretically give the same filter, the first of our methods is numerically more accurate and hence, renders a slightly better frequency response. Comparisons with both a linear programming approach and the eigenfilter formulation showed that the proposed methods are good in terms of both magnitude response and group delay variation.

The other new design approach was formulated to take the practical degradations due to both intersymbol interference and crosstalk into account. First, an analysis of these practical imperfections was done for each of the systems. Based on this analysis, the desired lowpass nature and Nyquist characteristic were considered together with the crosstalk that arises due to practical filters. The design procedure involves the optimization of an error function that is performed by a Quasi-Newton technique. The function proposed is based on (1) achieving a low stopband energy, (2) suppressing the mean-square intersymbol interference and (3) diminishing the crosstalk power. With an initial condition corresponding to a lowpass filter with an approximate or exact square root Nyquist frequency response, the resulting optimized filter leads to low intersymbol interference and crosstalk distortions.

The performance of the five transmultiplexers was compared for both minimax filters and the optimized filters. The intersymbol interference distortion is generally the lowest for system T3. This is due to the fact that for T3, a minimax design leads to filters that exactly satisfy the Nyquist criterion and the optimized design uses minimax filters as the initial condition. The normalized crosstalk power was observed to be generally lower for the QAM systems as compared to the VSB systems.

In comparing the design methods, we observed that lower intersymbol interference and crosstalk distortions with fewer filter coefficients are achieved by the optimized design when compared to minimax filters in the case of the one prototype systems. Therefore, the optimized design is preferred for T1, T2, T4 and T5. In the case of T3, both the minimax and the optimized design approaches are new. The advantage of the optimized design lies in using a crosstalk weight. This leads to a much lower crosstalk power than the minimax design for the same number of filter coefficients. Also, the resulting intersymbol interference distortion is very low although the Nyquist criterion is not exactly satisfied by the optimized design. When no crosstalk weight is applied, the optimized and minimax design approaches lead to a similar performance. For T3, there is a tradeoff between achieving a very low crosstalk distortion (optimized design) and exactly satisfying the Nyquist criterion (minimax design).

The complementary subband systems S1 to S5 achieve perfect reconstruction if the prototypes are bandlimited and the Nyquist criterion is satisfied. Therefore, the minimax designs for the transmultiplexers carry over to the subband complements. Moreover, the optimized designs without a crosstalk weight also carry over to the subband complements.

Finally, the issue of channel distortion in transmultiplexers was dealt with. In combating channel effects, the general principle is to use a single compensation filter that acts on the received composite signal prior to the action of the separation filter bank. This compensation filter was shown to have two components. One fixed component cancels crosstalk. The second component can be chosen to suppress intersymbol interference. Five choices for the second component were given. The first choice makes no attempt to control the intersymbol interference. Two other choices attempt to control either the time span of the intersymbol interference or the form of the input-output transfer function. The last two choices suppress the mean-square intersymbol interference. A performance evaluation involving a channel with a parabolic group delay showed that the last two choices achieve a low intersymbol interference distortion.

This investigation has led to new transmultiplexers and new filter design strategies that achieve an excellent performance. We anticipate that the new transmultiplexers will be important in practical data communication systems employing multicarrier transmission. Also, the new subband systems should be useful for speech coding applications.

7.3 Recommendations for Future Research

7.3.1 Adaptive Equalization of Channels

The configuration of the channel compensation filters was based on the assumption that the channel characteristic is known and is fixed for all time. However, the case of having a channel characteristic that is unknown and which varies with time should be investigated. Therefore, an adaptive equalizer that approximates the inverse of the channel characteristic is needed. The use of decision directed equalization is one possible approach [46][47].

7.3.2 Computational Complexity

A polyphase decomposition in conjunction with the use of a Fast Fourier Transform has been shown to substantially reduce the computational complexity of implementing the filter banks in [10][22]. In fact, this is an attractive feature of modulated filter banks. We anticipate that this technique is applicable to our new systems. The possibility of applying this technique to the new systems and comparing the transmultiplexers in terms of computational complexity is worth exploring.

7.3.3 Non-Uniform Modulated Filter Banks

This thesis has exclusively dealt with modulated filter banks in which each data signal is allocated exactly the same bandwidth. A pending problem involves relaxing the assumption of having equal bandwidth filter banks and synthesizing non-uniform banks. Recently, subband systems with an arbitrary number of bands with filter banks having non-uniform magnitude responses have been analyzed [48]. These subband systems differ from conventional structures in that the sampling rate compression/expansion factors are different in each band. However, there are necessary conditions on the sampling rate compression/expansion factors for aliasing cancellation [48]. First, analogous conditions for crosstalk cancellation in transmultiplexers with different sampling rate expansion/compression factors should be determined. Then, non-uniform modulated filter banks must be synthesized. The question of filter design should also emerge.

7.3.4 Subband Coding of Speech

Subband systems usually split the input speech into components that represent different frequency ranges. For individually coding each subband, the bit allocation can be weighted so that finer quantization is performed for the subbands that are perceptually more significant. Systems S4 and S5 decompose the speech into components representing different frequency ranges. The actual application of S4 and S5 to speech coding remains to be investigated. Systems S1, S2 and S3 are unusual in that modulated filter banks with repeated center frequencies are used. The potential advantages of S1, S2 and S3 for speech coding applications should be explored. Both scalar and vector quantization strategies should be considered in coding each subband.

Appendix A. Phase Factors in Relation to the Synthesis Procedure

Given the sum and difference criteria and the three sets of center frequencies, the sum of the phase factors $\alpha_k + \beta_k$ was confined to be a multiple of π for every terminal k. Here, we justify this choice based on a crosstalk analysis and design constraints. Consider the center frequencies in Set 1 which lead to system T1. For crosstalk cancellation between two signals sent at $\omega_k = 0$ and $\omega_l = 2\pi/N$, the condition that α_l and β_l be odd multiples of $\pi/2$ emerge if $n_k - p_l$ and $n_l - p_k$ are multiples of N. Then, $\alpha_l + \beta_l$ is a multiple of π. Considering either the sum or difference criterion reveals that the sum of the phase factors should be a multiple of π for each terminal.

Consider the frequencies of Set 2 which leads to system T2. In particular, we examine the crosstalk function relating two signals transmitted at $\omega_k = \omega_l = \pi/N$ (an end frequency). If the difference in the delay factors $n_k - p_l$ is an odd multiple of $N/2$, $\alpha_k + \beta_l$ should be a multiple of π and $\alpha_k - \beta_l$ should be an odd multiple of $\pi/2$ for cancelling the crosstalk. Combining these restrictions with those for either the sum or difference criterion and noting the conditions on the phase factors for the end frequencies leads us to confine the sum of the phase factors $\alpha_k + \beta_k$ and $\alpha_l + \beta_l$ to be a multiple of π. This restriction on the sum of the phase factors will then hold for every terminal.

In the case of the frequencies of Set 3, the arbitrary nature of the sum of the phase factors allows us to synthesize systems other than T4 and T5. The phase factors α_k and β_k of these systems will be different from those in T4 and T5. Also, the input-output transfer functions of these systems will differ from that of T4 and

T5 in that they will be a function of z^N as opposed to z^{2N} as in T4 and T5 (see Eq. (3.21)). Then, the condition for cancelling intersymbol interference is that $H^2(z)$ should be a Nyquist filter with an impulse response having zero crossings every Nth sample (except for a reference sample). This requires a minimum bandwidth of π/N (explained in Chapter 4) which corresponds to the maximum bandwidth allowed for the lowpass prototype $H(z)$. Hence, there is a conflict in the bandwidth constraints which renders an unsuitable design problem. By restricting $\alpha_k + \beta_k$ to be a multiple of π for every terminal, we encounter the feasible Nyquist design problem.

Appendix B. Derivation of Equation (3.9)

The crosstalk function specified by Eq. (3.8) is

$$T_{kl}(z^N) = \frac{1}{4} z^{-(n_k-p_l)} \sum_{i=0}^{N-1} \Big[[W^{(m-2p)(n_k-p_l)} e^{j(\alpha_k+\beta_l)} + e^{-j(\alpha_k+\beta_l)}] \\ W^{i(n_k-p_l)} H(zW^{-i+p}) H(zW^{-i-m+p}) \Big] \; . \qquad \text{(B.1)}$$

For notational convenience, let $n_k - p_l = s$, $m - 2p = r$ and $\alpha_k + \beta_l = \theta$. The crosstalk function is zero if

$$W^{rs} e^{j\theta} + e^{-j\theta} = 0 \qquad \text{(B.2)}$$

or equivalently

$$\begin{aligned} e^{j2\theta} &= -\frac{1}{W^{rs}} \\ &= -e^{j\frac{2\pi}{N} rs} \\ &= e^{j(\frac{2\pi}{N} rs + \pi)} \; . \end{aligned} \qquad \text{(B.3)}$$

This implies that

$$\theta = \pi \left[\frac{rs}{N} + \frac{1}{2} \right] \; . \qquad \text{(B.4)}$$

In terms of the original parameters, Eq. (B.4) becomes

$$\alpha_k + \beta_l = \pi \left[\frac{(m-2p)(n_k - p_l)}{N} + \frac{1}{2} \right] \; . \qquad \text{(B.5)}$$

Appendix C. Examination of the Crosstalk Function, Eq. (3.12)

For notational convenience, let $a = e^{j(\alpha_k+\beta_l)}$ and a^* be its complex conjugate. The first step in analyzing Eq. (3.12) is to substitute $\omega_k = (2\pi/N)q + \Delta\omega$ to get

$$\begin{aligned} 4z^{(n_k-p_l)}T_{kl}(z^N) = a \sum_{i=0}^{N-1} W^{i(n_k-p_l)} H^2(e^{-j\Delta\omega} z W^{-i+q}) \\ + a^* \sum_{i=0}^{N-1} W^{i(n_k-p_l)} H^2(e^{j\Delta\omega} z W^{-i-q}) \ . \end{aligned} \tag{C.1}$$

Note that q is an integer and $0 \leq \Delta\omega < 2\pi/N$. The limitations on $\Delta\omega$ are determined in order to fix the frequencies at which two signals can be transmitted without crosstalk. Let $e^{j\Delta\omega} = W^p$ where $-1 < p \leq 0$. Then,

$$\begin{aligned} 4z^{(n_k-p_l)}T_{kl}(z^N) = a \sum_{i=0}^{N-1} W^{i(n_k-p_l)} H^2(z W^{-i+q-p}) \\ + a^* \sum_{i=0}^{N-1} W^{i(n_k-p_l)} W^{-2q(n_k-p_l)} H^2(z W^{-i+q+p}) \ . \end{aligned} \tag{C.2}$$

It is desired to have the two terms in the above equation cancel each other.

Consider the case when $n_k - p_l$ is a multiple of N and $a = -a^*$. Then, the exponential indices of W in the arguments of $H^2(\cdot)$ of the two terms must differ by an integer to make the crosstalk zero. Therefore, p is fixed at either 0 or $-1/2$ thereby forcing the center frequencies to be multiples of π/N. Since $a = -a^*$, $\alpha_k + \beta_l$ is an odd multiple of $\pi/2$.

Suppose $n_k - p_l$ is an odd multiple of $N/2$. Then, we get

$$\begin{aligned} 4z^{(n_k-p_l)}T_{kl}(z^N) = a \sum_{i=0}^{N-1} (-1)^i H^2(z W^{-i+q-p}) \\ + a^* \sum_{i=0}^{N-1} (-1)^i H^2(z W^{-i+q+p}) \ . \end{aligned} \tag{C.3}$$

Algebraic substitution for the second term only yields

$$4z^{(n_k-p_l)}T_{kl}(z^N) = a\sum_{i=0}^{N-1}(-1)^i H^2(zW^{-i+q-p}) + a^*\sum_{i=-2p}^{N-1-2p}(-1)^i(-1)^{2p}H^2(zW^{-i+q-p})\ . \tag{C.4}$$

If $a = -a^*$, $2p$ must be an even integer for the two terms to cancel. Therefore, $p = 0$ and the center frequencies are multiples of $2\pi/N$. If $a = a^*$, $2p$ must be an odd integer for the two terms to cancel. Hence, $p = -1/2$ and the center frequencies are odd multiples of π/N. This development generates the various approaches as outlined in Section 3.4.2.

Appendix D. Two Band Systems: Repeated Center Frequencies

Consider two band systems with combining filters $A_k(z)$ and separation filters $B_k(z)$ for $k = 0$ and 1. The combining filters have parameters n_k and α_k. The separation filters have parameters p_k and β_k. The common center frequency is ω_c. For one prototype systems, we examine what possible values of ω_c are permissible for crosstalk cancellation. Consider the crosstalk function $T_{01}(z^2)$ given by

$$\begin{aligned} T_{01}(z^2) = \frac{1}{4} z^{-(n_0-p_1)} \Big[& e^{j(\alpha_0+\beta_1)} H^2(e^{-j\omega_c} z) \\ & + e^{-j(\alpha_0+\beta_1)} H^2(e^{j\omega_c} z) \\ & + 2\cos(\alpha_0 - \beta_1) H(e^{-j\omega_c} z) H(e^{j\omega_c} z) \\ & + (-1)^{-(n_0-p_1)} e^{j(\alpha_0+\beta_1)} H^2(-e^{-j\omega_c} z) \\ & + (-1)^{-(n_0-p_1)} e^{-j(\alpha_0+\beta_1)} H^2(-e^{j\omega_c} z) \\ & + (-1)^{-(n_0-p_1)} 2\cos(\alpha_0 - \beta_1) H(-e^{-j\omega_c} z) H(-e^{j\omega_c} z) \Big] \end{aligned} \tag{D.1}$$

Even if $\alpha_0 - \beta_1$ is an odd multiple of $\pi/2$ and two terms disappear, the sum of the other four terms should be zero. For this to happen, the arguments of H^2 must match. When $\omega_c \neq 0$ and $\omega_c \neq \pi$, the arguments match only if $e^{j\omega_c} = -e^{-j\omega_c}$ or $\omega_c = \pi/2$. This justifies the fact that two band systems can only use a repeated center frequency of $\pi/2$. The same arguments hold for the crosstalk function $T_{10}(z^2)$.

Appendix E. Constraints on the parameters l_0 and l_1

Let the zero crossing interval be K and the number of filter coefficients be $M = 2(l_0 + l_1) + 1$. The parameters l_0 and l_1 satisfy

$$l_0(K-1) \leq l_1 < l_0(K-1) + K \ . \tag{E.1}$$

Since $l_1 = (M-1)/2 - l_0$, the inequality reduces to

$$l_0 \leq \frac{M-1}{2K} < l_0 + 1 \ . \tag{E.2}$$

This new inequality is satisfied by a unique l_0 given by

$$l_0 = \left\lfloor \frac{M-1}{2K} \right\rfloor \ . \tag{E.3}$$

Then, l_1 is given by

$$l_1 = \frac{M-1}{2} - \left\lfloor \frac{M-1}{2K} \right\rfloor \ . \tag{E.4}$$

Appendix F. The Ratio l_1/l_0: Lower and Upper Bounds

This appendix derives lower and upper bounds for l_1/l_0 and shows how to fix the filter length $M = 2(l_0 + l_1) + 1$ to achieve these bounds. The zero crossing interval is K. The ratio l_1/l_0 is only finite for $l_0 \neq 0$ which is a reasonable assumption. If $l_0 = 0$, the filter length $M < 2K - 1$ thereby giving an impulse response with no zero crossings and hence, an insufficient length for an acceptable stopband attenuation.

F.1 Lower Bound

The lower bound for l_1/l_0 is given by the lefthand side of Eq. (E.1),

$$\frac{l_1}{l_0} \geq K - 1 \; . \tag{F.1}$$

The lower bound is achieved if and only if l_0 and l_1 are given by

$$\begin{aligned} l_0 &= \frac{M-1}{2K} \\ l_1 &= \frac{(M-1)(K-1)}{2K} \; . \end{aligned} \tag{F.2}$$

In this case, the filter length is of the form $M = 2l_0K + 1$ thereby giving an impulse response with the two end coefficients equal to zero.

If l_0 and l_1 are chosen as above, the system of equations $\mathbf{Df} = \mathbf{c}$ that solve for the coefficients of $F_0(z)$ can be decoupled into a reduced system of dimension l_0 and the additional equation $d(l_1)f_0(l_0) = 0$. Hence, $f_0(-l_0) = f_0(l_0) = 0$ thereby reducing the effective values of l_0 and M by 1 and 2 respectively. Such a choice of parameters gives results that are identical to the case when l_0 is reduced by 1.

F.2 Upper Bound

The upper bound for l_1/l_0 is obtained by examining the righthand side of Eq. (E.1),

$$\frac{l_1}{l_0} < K - 1 + \frac{K}{l_0} . \tag{F.3}$$

Since the minimum value of l_0 is 1, an upper bound is $2K - 1$. Achieving a ratio equal to a value of $2K - 2$ is possible if and only if $l_0 = 1$ and $l_1 = 2K - 2$. If $l_0 > 1$, the upper bound $K - 1 + K/l_0 \leq 2K - 2$ for every $K \geq 2$. Hence, $l_1/l_0 < 2K - 2$ for every $l_0 > 1$. The final conclusion is that for a given K, there exists only one filter length, namely, $M = 4K - 1$ that achieves the maximum value $l_1/l_0 = 2K - 2$.

Appendix G. Number of Exact Crosstalk Cancellations for a Specific Case

Consider a center frequency ω_c that is an even multiple of $2\pi/N$ (excluding 0 and π) in system T1 with N being a multiple of 4. For a signal sent at ω_c, exact crosstalk cancellation with other signals sent at odd multiples of $2\pi/N$ is achieved. Since there are $N/4$ frequencies that are odd multiples of $2\pi/N$ and two signals are sent at each of these frequencies, a total of $N/2$ crosstalk functions are exactly zero. In T1, there are a total of $(N-4)/4$ center frequencies that are even multiples of $2\pi/N$. The crosstalk between the signal sent at ω_c and one of the signals sent at other frequencies that are even multiples of $2\pi/N$ will be exactly zero depending on the delay factors. Furthermore, the crosstalk between the two signals sent at ω_c will be exactly zero. Now, we have an additional $(N-4)/4$ crosstalk functions that are exactly zero bringing the total to $(3N-4)/4$. In addition, the crosstalk between one of the signals sent at ω_c and the signals sent at 0 and π will be exactly zero depending on the delay factors. Depending on the signal sent at ω_c, the overall number of exact crosstalk cancellations is either $(3N-4)/4$ or $(3N+4)/4$.

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